Perturbative loop quantum gravity: Theory and observational implications

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Loop Quantum Gravity describe the gravitational field as $SU(2)$ non-Abelian gauge field using background independent methods. The canonical fields are so called Ashtekar variables $(A = A^i_a \tau_i dx^a, E = E^a_i \tau^i \partial_a)$ which take value in $\mathfrak{su}(2)$ and $\mathfrak{su}(2)^*$ algebras respectively and they fulfil the Poisson bracket

$$\{A^i_a(\mathbf{x}), E^b_j(\mathbf{y})\} = \gamma\kappa\delta^b_a\delta^i_j\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

where $\kappa = 8\pi G$ and γ is Barbero-Immirzi parameter. These variables are analogues of the vector potential and the electric field in electrodynamics. The Ashtekar variables are related with triad representation. In LQG gauge fields describe only spatial part Σ when time is treated separately.

To quantise this theory in the background independent way one introduces holonomies of connection A

$$h_e[A] = \mathcal{P} \exp \int_e A \quad \text{where 1-form} \quad A = \tau_i A^i_a dx^a$$

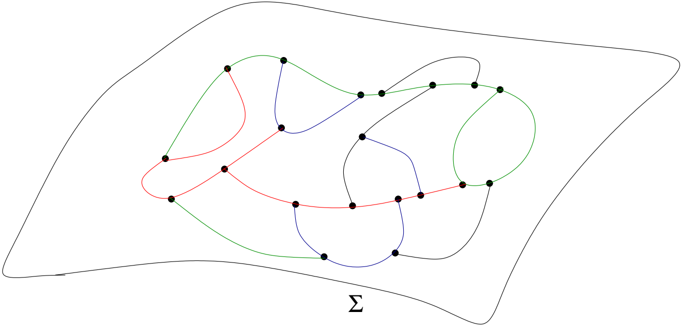
and conjugated fluxes

$$F^i_S[E] = \int_S dF^i \quad \text{where 2-form} \quad dF_i = \epsilon_{abc} E^a_i dx^b \wedge dx^c.$$

which are background independent observables. Here $\tau_i = -\frac{i}{2}\sigma_i$ where σ_i are Pauli matrices. Holonomies and fluxes fulfil the Poisson bracket

$$\{h_e[A], F^i_S[E]\} = \frac{\kappa\gamma}{4}\alpha(e,S)\tau^i h_e[A],$$

where $\alpha(e,S) = \pm 1$ in case $S \cap e \neq 0$ and $\alpha(e,S) = 0$ for $S \cap e = 0$.



Wave function in the Loop Quantum Gravity probes geometry only on the one dimensional submanifold called *spin network*

$$\Psi(A) := \psi(h_{e_1}(A), \ldots, h_{e_n}(A)) \in \text{Cyl}$$

where $\{e_1, \ldots, e_n\}$ are edges of the spin network.

Hamiltonian

Hamiltonian takes the form of a liner combination of the constraints

$$H_G = \int_\Sigma d^3\mathbf{x}\,(N^i G_i + N^a C_a + NS).$$

Spatial diffeomorphisms constraint:

$$C_a = E_i^b F_{ab}^i - (1-\gamma^2) K_a^i G_i.$$

Gauss constraint:

$$G_i = D_a E_i^a = \partial_a E_i^a + \epsilon_{ijk} A_a^j E_k^a.$$

Scalar constraint:

$$S = \frac{E_i^a E_j^b}{\sqrt{|\det E|}} \left[\varepsilon^{ij}{}_k F_{ab}^k - 2(1+\gamma^2) K_{[a}^i K_{b]}^j\right]$$

where $F = dA + \frac{1}{2}[A, A]$.

In the above expressions $\gamma K^i_a = A^i_a - \Gamma^i_a$ where Γ^i_a is a spin connection which can be expressed in terms of the field E. Namely

$$\Gamma^i_a = -\frac{1}{2}\epsilon^{ij}{}_k e^b_j \left(\partial_{[a} e^k_{b]} + \delta^{kl}\delta_{ms} e^c_l e^m_a \partial_b e^s_c\right)$$

where

$$e^i_a = \frac{1}{2}\frac{\epsilon_{abc}\epsilon^{ijk}E^b_j E^c_k}{\sqrt{|\det(E)|}} \quad \text{and} \quad e^a_i = \frac{\mathrm{sgn}(\det(E))E^a_i}{\sqrt{|\det(E)|}}.$$

In this way the theory is fully defined on the classical level. Later we can write

$$E^a_i = \sqrt{|\det q|}e^a_i$$

where $q_{ab} = e^i_a e^j_b \delta_{ij}$ what bring us to the expression

$$\det(q)q^{ab} = E^a_i E^b_j \delta^{ij}.$$

Therefore the field Ecan be interpreted in terms of the metric field q_{ab}.

Perturbations

We perturb basic variables around a spatially flat FRW background

$$\begin{aligned} E^a_i &= \bar{E}^a_i + \delta E^a_i, \\ A^i_a &= \bar{A}^i_a + \delta A^i_a. \end{aligned}$$

Background components have the following form

$$\begin{aligned} \bar{E}^a_i &= \bar{p}\delta^a_i, \\ \bar{A}^i_a &= \gamma \bar{k} \delta^i_a, \end{aligned}$$

where $\bar{p} = a^2$ and $\bar{k} = \dot{\bar{p}}/2\bar{p}$. Perturbations can be split for the

- scalar (coupled with scalar field - fairly complicated system)
- vector (simple but less interesting)
- tensor (gravitational waves - relatively simple)

Gravitational waves

Tensor perturbations are introduced as follows

$$\begin{aligned} g_{00} &= -N^2 + q_{ab}N^a N^b = -\bar{N} = -a^2, \\ g_{0a} &= q_{ab}N^b = 0, \\ g_{ab} &= q_{ab} = a^2[\delta_{ab} + h_{ab}], \end{aligned}$$

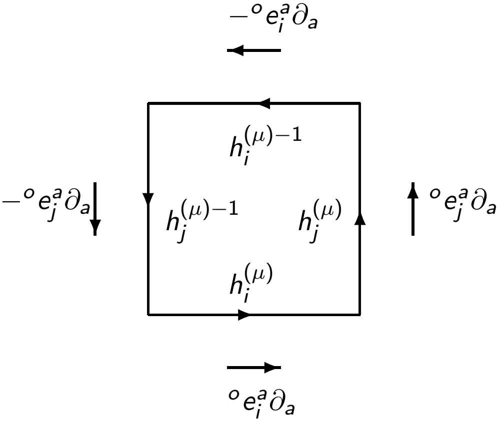
with conditions $h^a_a = \partial_a h^a_b = 0$ and $|h_{ab}| \ll 1$. In the TT gauge $h^1_1 = -h^2_2 = h_\oplus$ and $h^1_2 = h^2_1 = h_\otimes$. Second order perturbed Hamiltonian takes the form

$$\begin{aligned} H_G = \frac{1}{16\pi G}\int_\Sigma d^3x \bar{N}\Bigg[&-6\bar{k}\sqrt{\bar{p}} - \frac{\bar{k}^2}{2\bar{p}^{3/2}}(\delta E^c_j \delta E^d_k \delta^k_c \delta^j_d) + \sqrt{\bar{p}}(\delta K^j_c \delta^k_d \delta^c_k \delta^d_j) \\ &- \frac{2\bar{k}}{\sqrt{\bar{p}}}(\delta E^c_j \delta K^j_c) + \frac{1}{\bar{p}^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_e E^c_j \partial_f E^d_k)\Bigg] \end{aligned}$$

where $\delta E^a_i = -\frac{1}{2}\bar{p}h^a_i$ and $\delta K^i_a = \frac{1}{2}\left[\dot{h}^i_a + \bar{k}h^i_a\right]$.

Holonomy corrections

$$h^{(\mu)}_{\Box_{ij}} = h^{(\mu)}_i h^{(\mu)}_j h^{(\mu)-1}_i h^{(\mu)-1}_j.$$



It is straightforward to show that the field strength can be expressed as

$$F_{ab}^{k} = -2 \lim_{\text{Ar}\to 0} \frac{\text{tr}\left[\tau_k \left(h_{\Box_{ij}}^{(\mu)} - \mathbb{I}\right)\right]}{\mu^2} {}^{o}\omega_a^i {}^{o}\omega_b^j.$$

The trace in this equation can be explicitly calculated

$$\text{tr}\left[\tau_k \left(h_{\Box_{ij}}^{(\mu)} - \mathbb{I}\right)\right] = -\frac{\epsilon_{kij}}{2} \sin^2\left(\mu\gamma\bar{k}\right).$$

In Loop Quantum Gravity the limit $\text{Ar} \to 0$ does not exist because of existence of the area gap. The area gap corresponds to the minimal quanta of area $\Delta = 2\sqrt{3}\pi\gamma l_{\text{Pl}}^2$. Classically

$$F_{ab}^{k} = \gamma^2 \bar{k}^2 \epsilon_{kij} {}^{o}\omega_a^i {}^{o}\omega_b^j.$$

Hamiltonian with holonomy corrections

Holonomy corrections are introduced by the replacement

$$\bar{k} \to \frac{\sin \bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}$$

in the classical expressions. Here

$$\bar{\mu} = \sqrt{\frac{\Delta}{\bar{p}}} \quad \text{where} \quad \Delta = 2\sqrt{3}\pi\gamma l_{\text{Pl}}^2.$$

Effective second order Hamiltonian takes the form

$$\begin{aligned} H_G^{\text{phen}} &= \frac{1}{16\pi G}\int_\Sigma d^3x\bar{N}\left[-6\sqrt{\bar{p}}\left(\frac{\sin \bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2\right. \\ &- \frac{1}{2\bar{p}^{3/2}}\left(\frac{\sin \bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2(\delta E_j^c\delta E_k^d\delta_c^k\delta_d^j) + \sqrt{\bar{p}}(\delta K_c^j\delta K_d^k\delta_k^c\delta_j^d) \\ &- \left.\frac{2}{\sqrt{\bar{p}}}\left(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\right)(\delta E_j^c\delta K_c^j) + \frac{1}{\bar{p}^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_e E_j^c\partial_f E_k^d)\right]. \end{aligned}$$

Gravitational waves with holonomy correction

Based on the Hamilton equations

$$\begin{aligned} \delta \dot{E}^a_i &= \left\{ \delta E^a_i, H^{\text{phen}}_G + H_m \right\}, \\ \delta \dot{K}^i_a &= \left\{ \delta K^i_a, H^{\text{phen}}_G + H_m \right\}, \end{aligned}$$

where $\delta E^a_i = -\frac{1}{2}\bar{p}h^a_i$ and $\delta K^i_a = \frac{1}{2}\left[\dot{h}^i_a + \left(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\right) h^i_a\right]$ we obtain

$$\ddot{h}^i_a + 2\bar{k}\dot{h}^i_a - \nabla^2 h^i_a + T_Q h^i_a = 16\pi G \Pi^i_{Qa}$$

where

$$\begin{aligned} T_Q &= -2\left(\frac{\bar{p}}{\bar{\mu}}\frac{\partial\bar{\mu}}{\partial\bar{p}}\right)\bar{\mu}^2\gamma^2\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^4, \\ \Pi^i_{Qa} &= \left[\frac{1}{3V_0}\frac{\partial\bar{H}_{\text{m}}}{\partial\bar{p}}\left(\frac{\delta E^c_j\delta^j_a\delta^i_c}{\bar{p}}\right)\cos 2\bar{\mu}\gamma\bar{k} + \frac{\delta H_m}{\delta(\delta E^a_i)}\right] \end{aligned}$$

are quantum holonomy corrections.

Matter

We consider homogeneous scalar field with the Hamiltonian

$$H_m = \int_\Sigma d^3\mathbf{x}\,\bar{N}\left(\frac{1}{2}\frac{\pi_\phi^2}{\sqrt{|\det E|}} + \sqrt{|\det E|}V(\phi)\right),$$

where up to the second order

$$\begin{aligned}
\sqrt{\det E} &= \bar{p}^{\frac{3}{2}}\left[1+\frac{1}{2\bar{p}}\delta^i_a\delta E^a_i - \frac{1}{4\bar{p}^2}\delta^i_a\delta E^a_j\delta^j_b\delta E^b_i + \frac{1}{8\bar{p}^2}\delta^i_a\delta E^a_i\delta^j_b\delta E^b_j\right],\\
\frac{1}{\sqrt{\det E}} &= \frac{1}{\bar{p}^{\frac{3}{2}}}\left[1-\frac{1}{2\bar{p}}\delta^i_a\delta E^a_i + \frac{1}{4\bar{p}^2}\delta^i_a\delta E^a_j\delta^j_b\delta E^b_i + \frac{1}{8\bar{p}^2}\delta^i_a\delta E^a_i\delta^j_b\delta E^b_j\right].
\end{aligned}$$

However, since $\delta^{ab}h_{ab} = 0 \Rightarrow \delta^i_a\delta E^a_i = 0$ above expansion simplify. Then

$$H_m = \bar{H}_m + \frac{1}{4}\int_\Sigma d^3\mathbf{x}\,\frac{\bar{N}}{\sqrt{\bar{p}}}\left(\frac{1}{2}\frac{\pi_\phi^2}{\bar{p}^3} - V(\phi)\right)\delta^i_a\delta E^a_j\delta^j_b\delta E^b_i + \mathcal{O}(E^3).$$

Expressions for the quantum holonomy corrections simplify to

$$
\begin{aligned}
T_Q &= \frac{8\pi G}{3}\frac{\bar{p}\rho^2}{\rho_c}, \\
\Pi^i_{Qa} &= \Pi_Q h^i_a = \frac{1}{2}\bar{p}\frac{\rho}{\rho_c}\left(2V-\rho\right)h^i_a .
\end{aligned}
$$

Now equation for the tensor modes simplify to

$$
\frac{d^2}{d\eta^2}h^i_a + 2\bar{k}\frac{d}{d\eta}h^i_a - \nabla^2 h^i_a + \tilde{T}_Q h^i_a = 0
$$

where we have defined the total holonomy correction

$$
\tilde{T}_Q = T_Q - 16\pi G \Pi_Q = 16\pi G \bar{p}\frac{\rho}{\rho_c}\left(\frac{2}{3}\rho - V\right).
$$

Introducing a new variable

$$u = \frac{a h_{\oplus}}{\sqrt{16\pi G}} = \frac{a h_{\otimes}}{\sqrt{16\pi G}}$$

and performing the Fourier transform

$$u(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} u(\tau, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

we can rewrite the equation

$$\frac{d^2}{d\eta^2} h_a^i + 2\bar{k} \frac{d}{d\eta} h_a^i - \nabla^2 h_a^i + \tilde{T}_Q h_a^i = 0$$

to the form

$$\frac{d^2}{d\tau^2} u(\tau, \mathbf{k}) + [k^2 + m_{\text{eff}}^2] u(\tau, \mathbf{k}) = 0$$

where $k^2 = \mathbf{k} \cdot \mathbf{k}$ and

$$m_{\text{eff}}^2 = \tilde{T}_Q - \frac{a''}{a}.$$

Free scalar field

Energy density of the free scalar field has the form

$$\rho = \frac{1}{2}\frac{\pi_\phi^2}{\bar{p}^3}$$

and solution of the Effective Friedmann equation (effective background equation)

$$\left(\frac{1}{2\bar{p}}\frac{d\bar{p}}{dt}\right)^2 = \frac{\kappa}{3}\rho\left(1-\frac{\rho}{\rho_c}\right)$$

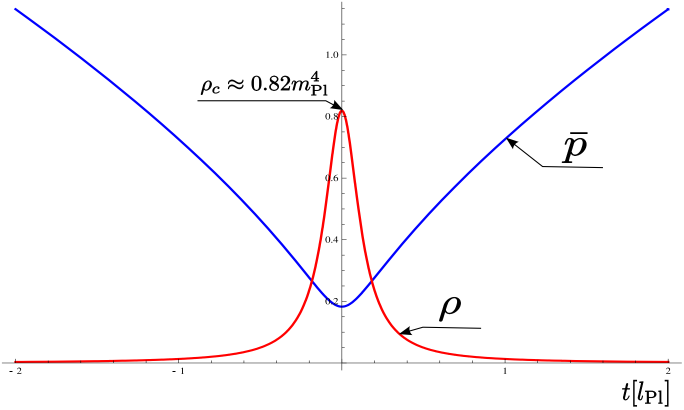
is the following

$$\bar{p}(t) = \left(A + Bt^2\right)^{1/3}$$

where

$$A = \frac{1}{6}\kappa\pi_\phi^2\gamma^2\Delta \quad , \quad B = \frac{3}{2}\kappa\pi_\phi^2 \quad \text{and} \quad \rho_c = \frac{\sqrt{3}}{16\pi^2\gamma^3 l_{\text{Pl}}^4}.$$

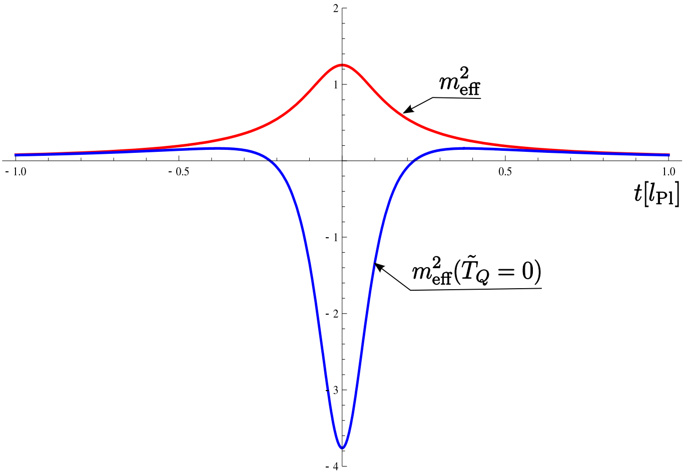
Bounce



For other bounces in the effective LQC see [1]

[1] J. Mielczarek, T. Stachowiak and M. Szydlowski, "Exact solutions for Big Bounce in loop quantum cosmology," Phys. Rev. D **77** (2008) 123506 [arXiv:0801.0502 [gr-qc]].

$$m_{\text{eff}}^2 = \frac{\kappa^2\pi_\phi^2}{4}\frac{\left(t^2+\frac{1}{9}\gamma^2\Delta\right)}{(A+Bt^2)^{5/3}} \geq 0$$



Multi-fluid potential

Recently, multi-fluid potential in the form [2]

$$V(\phi) = \frac{V_0}{\cosh^2\left[\sqrt{6\pi G(1+w)}\phi\right]}$$

where $V_0 = \rho_c(1-w)/2$ was introduced. Scalar field with this potential mimic barotropic matter with equation of state $p = w\rho$ ($w = \text{const}$). In this case

$$\begin{aligned} m^2_{\text{eff}} &= \bar{p}_c \frac{3}{16}\kappa^2\rho_c^2(1+w)^2(3w-1)(1+6\pi G\rho_c(1+w)^2t^2)^{\alpha} \times \\ &\times \left\{t^2 + \frac{4}{3}\frac{\Delta\gamma^2}{(1+w)(1-3w)}\left[1-\frac{32}{9}\frac{(1+3w)}{(1+w)}\right]\right\} \end{aligned}$$

where

$$\alpha = -(2/3)(2+3w)/(1+w).$$

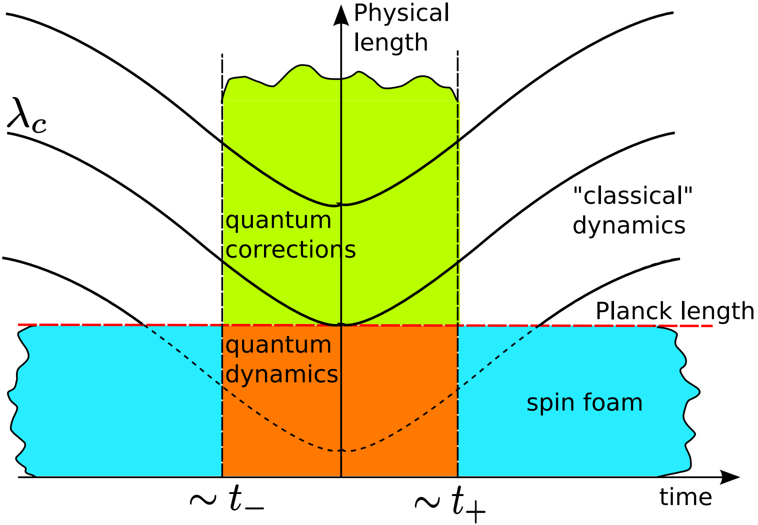
[2] J. Mielczarek, "Multi-fluid potential in the loop cosmology," arXiv:0809.2469 [gr-qc].

Why multi-fluid potential can be useful:

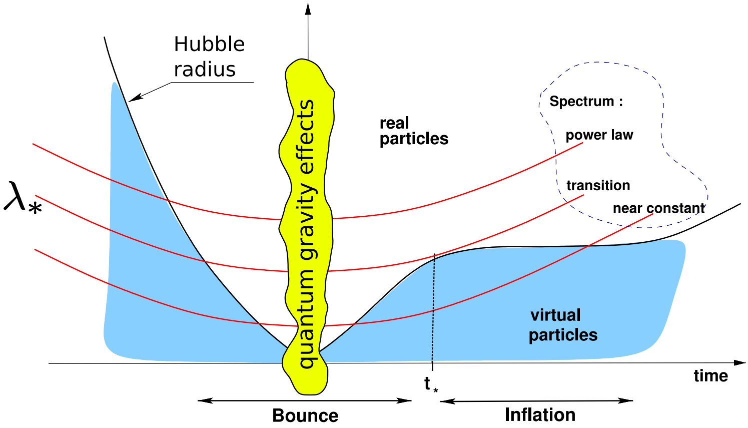
- Effective equations are solved analytically.
- For the model with $w = 0$ we obtain scale-invariant power spectrum of perturbations. Therefore we do not need inflation!
- Field ϕ is monotonic function of time t. This indicate that it can be treated as a well defined internal time for these models.
- Production of non-Gaussianities (currently under investigations). Field with the multi-fluid potential is non-Gaussian (modes are interacting one another).

$$\langle \phi_{\mathbf{k_1}} \phi_{\mathbf{k_2}} \phi_{\mathbf{k_3}} \rangle = \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) P_3(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) \neq 0$$

Therefore large scale non-Gaussian effects can be produced. This phenomenon can be potentially probed with the astronomical observations. Current bounds from the CMB nonlinearities can constraint quantum cosmology models!


Physical
length
λ_c
quantum
corrections
"classical"
dynamics
Planck length
quantum
dynamics
spin foam
$\sim t_-$
$\sim t_+$
time

Bounce+Inflation

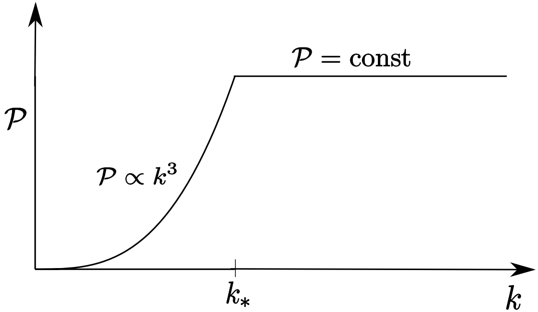


Bounce+Inflation model can solve the low CMB multipoles problem. [3]

[3] J. Mielczarek, "Gravitational waves from the Big Bounce," JCAP **0811** (2008) 011 [arXiv:0807.0712 [gr-qc]].

Toy model

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{A}\Theta(k - k_*) + \mathcal{A}\Theta(k_* - k)\left(\frac{k}{k_*}\right)^3$$



After short calculations we obtain expression

$$\mathcal{C}_l = \mathcal{C}_l^{\text{inf}} \left(1 + \delta_l(x_*)\right)$$

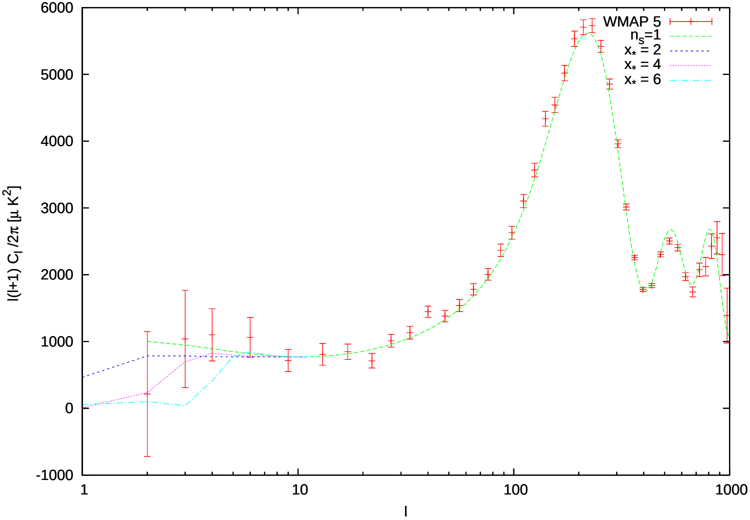
where

$$\mathcal{C}_l^{\text{inf}} = \frac{\mathcal{A}}{25}$$

and

$$\delta_l(x_*) = l(l+1)\left\{ j_l^2(x_*) - j_l(x_*)j_{l+1}(x_*) - \frac{\pi 4^{-l-1} x_*^{2l} {}_1F_2(l; l+3/2, 2l+2; -x_*^2)}{l\Gamma^2(l+3/2)} \right\}$$

Here $x_* = D_{LSS} k_*$ where D_{LSS} is the distance to the Last Scattering Shell.


WMAP 5
n_s=1
x_* = 2
x_* = 4
x_* = 6
6000
5000
4000
3000
2000
1000
0
-1000
1
10
100
1000
l
l(l+1) C_l /2π [μ K^2]

Summary and outlook

- Studies of perturbations give us potential way to test quantum gravity effects.
- Big bounce phase can explain low multipoles suppression observed in the CMB.
- We do not need inflation in the model with the multi-fluid potential ($w = 0$).
- Non-Gaussian effects, CMB polarization by the relic gravitational waves - work in progress.
- Scalar modes with holonomy corrections - work in progress.
- Gauge invariant approach

LQG & LQC

A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21** (2004) R53 [arXiv:gr-qc/0404018].

M. Bojowald, "Loop quantum cosmology," Living Rev. Rel. **11** (2008) 4 [http://www.livingreviews.org/lrr-2008-4].

Perturbations

M. Bojowald, G. M. Hossain, M. Kagan and S. Shankaranarayanan, "Anomaly freedom in perturbative loop quantum gravity," Phys. Rev. D **78** (2008) 063547 [arXiv:0806.3929 [gr-qc]].

M. Bojowald, G. M. Hossain, M. Kagan and S. Shankaranarayanan, "Gauge invariant cosmological perturbation equations with corrections from loop quantum gravity," arXiv:0811.1572 [gr-qc].

Gravitational waves in LQC

J. Mielczarek and M. Szydlowski, "Relic gravitons as the observable for Loop Quantum Cosmology," Phys. Lett. B **657** (2007) 20 [arXiv:0705.4449 [gr-qc]].

M. Bojowald and G. M. Hossain, "Loop quantum gravity corrections to gravitational wave dispersion," Phys. Rev. D **77** (2008) 023508 [arXiv:0709.2365 [gr-qc]].

J. Mielczarek and M. Szydlowski, "Relic gravitons from super-inflation," arXiv:0710.2742 [gr-qc].

E. J. Copeland, D. J. Mulryne, N. J. Nunes and M. Shaeri, "The gravitational wave background from super-inflation in Loop Quantum Cosmology," arXiv:0810.0104 [astro-ph].

J. Mielczarek, "Gravitational waves from the Big Bounce," JCAP **0811** (2008) 011 [arXiv:0807.0712 [gr-qc]].