Stellar and extragalactic astronomy I - Problem Set 4
9.12.2010, Thursday, 10.45-12.15

1. **Isothermal sphere - continuation.**

a) Consider Boltzmann distribution function in the form

$$f(\mathbf{v},\mathbf{x}) = C\exp\left(-\frac{E(\mathbf{v},\mathbf{x})}{kT}\right), \tag{1}$$

where

$$E(\mathbf{v},\mathbf{x}) = \frac{m\mathbf{v}^2}{2} + m\Phi(\mathbf{x}). \tag{2}$$

Show that the corresponding distribution of matter can be expressed as follows

$$\rho(\mathbf{x}) = \rho_* \exp\left(-\frac{m\Phi(\mathbf{x})}{kT}\right), \tag{3}$$

where $\rho_* = Cm\left(\frac{2\pi kT}{m}\right)^{3/2}$.

b) Assuming spherical symmetry, show that Poisson equation for the matter distribution (3) is

$$\frac{d^2}{dr^2}\Phi(r) + \frac{2}{r}\frac{d}{dr}\Phi(r) = 4\pi G\rho_* \exp\left(-\frac{m\Phi(r)}{kT}\right), \tag{4}$$

where $r = |\mathbf{x}|$. Show that function

$$\Phi(r) = -\frac{kT}{m}\ln\left(\frac{kT}{2\pi G\rho_* m r^2}\right) \tag{5}$$

is solution of equation (4). This solution is known as the *singular isothermal sphere* (SIS). Show that potential (5) leads to the following distribution of matter

$$\rho(r) = \frac{kT}{2\pi Gm}\frac{1}{r^2}. \tag{6}$$

c) Show that equipartition theorem leads to relation $\frac{kT}{m} = \sigma^2$. Assuming ideal gas law show that pressure $p = \sigma^2\rho$.

2. **Light deflection.** As in the case of the prism, light rays are deflected when they pass through a gravitational field. The deflection angle is the integral along the light path of the gradient of Φ perpendicular to the light path, i.e.

$$\vec{\alpha}(d) = \frac{2}{c^2}\int dl \vec{\nabla}_\perp \Phi, \tag{7}$$

where d is the impact parameter of the unperturbed light ray, which is the shortest distance from the center of mass. Calculate the deflection angle of light in the presence of a mass distribution of the isothermal sphere (6). Assume that the photon path can be approximated by a straight and infinite line. Assume that the mass distribution of an elliptic galaxy can be well approximated by the SIS model, with a typical dispersion of velocity $\sigma = 300$ km/s. Calculate the light deflection angle originating at a distant point source for $d = 10$ kpc.

3. **Galaxies in expanding Universe.** Let us consider isotropic and homogeneous model of the Universe. Physical distance between two object is given by $\vec{r} = a\vec{x}$, where a is a scale factor and $\vec{x}$ is the coordinate distance. Then $\dot{\vec{r}} = H\vec{r} + \vec{v}$, where $H \equiv \dot{a}/a$ is the Hubble factor and $\vec{v}$ is called *peculiar velocity*. Show that $\ddot{\vec{r}} = \dot{\vec{v}} + H\vec{v} - qH^2\vec{r}$, where $q \equiv -\frac{\ddot{a}a}{\dot{a}^2}$ is deceleration parameter. The Newton equation is given by $m\dot{\vec{v}} = \vec{F}$. Consider circular motion of a particle around a central mass M. Show that equation of motion is given by

$$m\frac{d^2r}{dt^2} = \frac{L^2}{mr^3} - \frac{GMm}{r^2} - qH^2mr, \tag{8}$$

where L is angular momentum. Assume the present values $H = H_0 = 70.3$ km/s/Mpc and $q = q_0 = -0.62$. Consider circular motion of a star at radius $r = 12$ kpc around the center of Milky Way Galaxy, then $M \approx 10^{12}M_\odot$. Compare the force due to the cosmic expansion with the gravitational force. Is the effect of present cosmic expansion relevant for galactic dynamics?

4. **Jeans equation once again.** Show that Jeans equation (2) from the Problem Set 3 can be expressed as follows

$$v_c^2 = -\sigma_{rr}^2(\gamma + \kappa + 2\beta), \tag{9}$$

where v_c is circular velocity, $\gamma \equiv \frac{d\ln\rho}{d\ln r}$, $\kappa \equiv \frac{d\ln\sigma_t^2}{d\ln r}$ and $\beta \equiv 1 - \frac{\sigma_t^2}{\sigma_{rr}^2}$ is velocity dispersion anisotropy.

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