Stellar and extragalactic astronomy I - Problem Set 3
2.12.2010, Thursday, 10.45-12.15

1. **Isothermal sphere.** Radial Jeans equation in spherical coordinates takes the following form:

$$\nu\frac{\partial \bar{v}_r}{\partial t}+\nu\left(\bar{v}_r\frac{\partial \bar{v}_r}{\partial r}+\frac{\bar{v}_\theta}{r}\frac{\partial \bar{v}_r}{\partial \theta}+\frac{\bar{v}_\phi}{r\sin\theta}\frac{\partial \bar{v}_r}{\partial \phi}\right)+\frac{\partial}{\partial r}\left(\nu\sigma^2_{rr}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\nu\sigma^2_{r\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\nu\sigma^2_{r\phi}\right)$$
$$+\frac{\nu}{r}\left[2\sigma^2_{rr}-\left(\sigma^2_{\theta\theta}+\sigma^2_{\phi\phi}+\bar{v}^2_\theta+\bar{v}^2_\phi\right)+\sigma^2_{r\theta}\cot\theta\right]=-\nu\frac{\partial\Phi}{\partial r}. \quad (1)$$

a) Let us consider spherically symmetric, steady-state system of gravitationally interacting particles. Steady-state hydrodynamic equilibrium implies $\bar{v}_r=0$ and $\frac{\partial\bar{v}_r}{\partial t}=0$. Spherical symmetry implies $\bar{v}_\theta=\bar{v}_\phi=0$, $\sigma^2_{r\theta}=\sigma^2_{r\phi}=\sigma^2_{\theta\phi}=0$ and $\sigma^2_{\theta\theta}=\sigma^2_{\phi\phi}\equiv\sigma^2_t$. Show that, with these assumptions, equation (1) simplifies to

$$\frac{1}{\nu}\frac{d}{dr}(\nu\sigma^2_{rr})+2\frac{(\sigma^2_{rr}-\sigma^2_t)}{r}=-\frac{d\Phi}{dr}. \quad (2)$$

b) Taking spherical symmetry into account, show that Poisson equation implies

$$\frac{d\Phi}{dr}=\frac{GM(r)}{r^2}\quad\text{where}\quad M(r)=4\pi\int_0^r\rho(r')r'^2dr'. \quad (3)$$

c) We define the anisotropy parameter β as

$$\beta\equiv1-\frac{\sigma^2_t}{\sigma^2_{rr}}. \quad (4)$$

Based on (2) and (3) show that

$$M(r)=-\frac{r\sigma^2_{rr}}{G}\left[\frac{d\ln\nu}{d\ln r}+\frac{d\ln\sigma^2_{rr}}{d\ln r}+2\beta(r)\right]. \quad (5)$$

d) Let us assume that the velocity dispersion tensor is isotropic, then $\sigma^2_{rr}=\sigma^2_t=\sigma^2$. Starting from equation (2) and making the additional assumption of *isothermality* (taking the velocity dispersion σ to be independent of radius) show that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\rho\right)=-\frac{4\pi G\rho}{\sigma^2}, \quad (6)$$

where density of matter $\rho=m\nu$. Verify that

$$\rho(r)=\frac{\sigma^2}{2\pi Gr^2} \quad (7)$$

is solution of equation (6). This solution describes a model known as the *singular isothermal sphere*. Show that the corresponding mass function is given by $M(r)=\frac{2\sigma^2 r}{G}$. Discuss the limits $r\to0$ and $r\to\infty$ for $\rho(r)$ and $M(r)$.

e) Consider circular motion (constant r) of a test particle (e.g. star) in spherically symmetric matter distribution. Show that

$$\frac{v^2_c}{r}=\frac{d\Phi}{dr}\quad\text{and}\quad v^2_c=\frac{GM(r)}{r}, \quad (8)$$

where v_c is circular velocity. Show that for the matter distribution (7) the circular velocity can be expressed as follows

$$v_c=\sqrt{2}\sigma. \quad (9)$$

How this result is related with the problem of flat galactic rotation curves? Compare the obtained result with the velocity curves expected for $\rho=$ const and $M=$ const. Find exemplary plots of galactic rotation curves and discuss their features.

f) Let us consider a model of galactic halo composed of hypotetical particles called *axions*, $m_\mathrm{a}=10^{-3}$ eV. For the Milky Way Galaxy $v_c\approx200$km/s. Based on this, and the equipartition theorem estimate temperature of the axions. Are these axions relativistic? Show that the following expressions are fulfilled:

$$\rho(r)=\frac{kT}{2\pi Gm_\mathrm{a}}\frac{1}{r^2}\quad\text{and}\quad M(r)=\frac{2kT}{m_\mathrm{a}G}r. \quad (10)$$

Estimate density of axions in the lecture room. Find $M(20\text{kpc})$ and express the result in terms of $M_\odot$.

Jakub Mielczarek