Stellar and extragalactic astronomy I - Problem Set 2
25.11.2010, Thursday, 9.00-10.30

1. **Continuity equation.** Based on the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x^i}\frac{\partial f}{\partial v_i} = 0, \tag{1}$$

derive continuity equation

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \bar{v}_i) = 0, \quad \text{where} \tag{2}$$

$$\nu(\mathbf{x}) \equiv \int d^3v f(\mathbf{v},\mathbf{x}) \quad \text{and} \quad \bar{v}_i(\mathbf{x}) \equiv \frac{1}{\nu(\mathbf{x})}\int d^3v v_i f(\mathbf{v},\mathbf{x}). \tag{3}$$

Take into account that $\lim_{|\mathbf{x}|\to\infty} f(\mathbf{v},\mathbf{x}) = 0$ and use the following relation

$$\int_V \mathbf{g}\cdot\nabla_v f d^3v = \int_{\Sigma=\partial V} f\mathbf{g}\cdot\mathbf{dS} - \int_V f\nabla_v\cdot\mathbf{g}d^3v. \tag{4}$$

2. **Jeans equations.** Based on the collisionless Boltzmann equation (1) derive Jeans equations

$$\nu\frac{\partial \bar{v}_j}{\partial t} + \bar{v}_i\nu\frac{\partial \bar{v}_j}{\partial x_i} = -\nu\frac{\partial\Phi}{\partial x_j} - \frac{\partial}{\partial x_i}(\nu\sigma^2_{ij}). \tag{5}$$

$$\text{acceleration} = \begin{matrix}\text{gravity}\\ \text{force}\end{matrix} + \begin{matrix}\text{gradient}\\ \text{"pressure"}\end{matrix}$$

Take into account the following defnitions

$$\nu(\mathbf{x}) \equiv \int d^3v f(\mathbf{v},\mathbf{x}) \quad \text{(concentration)}, \tag{6}$$

$$\bar{v}_i(\mathbf{x}) \equiv \frac{1}{\nu(\mathbf{x})}\int d^3v v_i f(\mathbf{v},\mathbf{x}) \quad \text{(mean velocity)}, \tag{7}$$

$$\sigma^2_{ij} \equiv \overline{(v_i-\bar{v}_i)(v_j-\bar{v}_j)} = \overline{v_iv_j} - \bar{v}_i\bar{v}_j \quad \text{(velocity-dispersion tensor)}. \tag{8}$$

2. **Boltzmann and Jeans equations in spherical coordinates.** Show that in spherical coordinates the collisionless Boltzmann equation is

$$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial f}{\partial r} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{v}_r\frac{\partial f}{\partial v_r} + \dot{v}_\theta\frac{\partial f}{\partial v_\theta} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} = 0, \tag{9}$$

where

$$\dot{r} = v_r, \tag{10}$$

$$\dot{\theta} = \frac{v_\theta}{r}, \tag{11}$$

$$\dot{\phi} = \frac{v_\phi}{r\sin\theta}, \tag{12}$$

$$\dot{v}_r = \frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial\Phi}{\partial r}, \tag{13}$$

$$\dot{v}_\theta = \frac{v_\phi^2\cot\theta - v_rv_\theta}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \tag{14}$$

$$\dot{v}_\phi = \frac{-v_\phi v_r - v_\phi v_\theta\cot\theta}{r} - \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}. \tag{15}$$

With use this, derive the radial Jeans equation

$$\nu\frac{\partial\bar{v}_r}{\partial t} + \nu\left(\bar{v}_r\frac{\partial\bar{v}_r}{\partial r} + \frac{\bar{v}_\theta}{r}\frac{\partial\bar{v}_r}{\partial\theta} + \frac{\bar{v}_\phi}{r\sin\theta}\frac{\partial\bar{v}_r}{\partial\phi}\right) + \frac{\partial}{\partial r}\left(\nu\sigma^2_{rr}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\nu\sigma^2_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\nu\sigma^2_{r\phi}\right)$$
$$+\frac{\nu}{r}\left[2\sigma^2_{rr} - \left(\sigma^2_{\theta\theta} + \sigma^2_{\phi\phi} + \bar{v}^2_\theta + \bar{v}^2_\phi\right) + \sigma^2_{r\theta}\cot\theta\right] = -\nu\frac{\partial\Phi}{\partial r}. \tag{16}$$

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