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# Relic gravitons as the observable for loop quantum cosmology

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**Abstract**

In this Letter we investigate tensor modes of perturbations in the universe governed by loop quantum cosmology. We derive the equation for tensor modes and investigate numerically effects of quantum corrections. This investigation reveals that the region of super-adiabatic amplification of tensor modes is smaller in comparison with the classical case. Neglecting quantum corrections to the equation for tensor modes and holding underlying loop dynamics we study analytically the creation of gravitons. We calculate the power spectrum of tensor perturbations during the super-inflationary phase induced by loop quantum gravity. The main result obtained is the spectrum of gravitons, produced in the transition from the quantum to classical regime of the Universe. Obtained spectrum is characterized by a hard branch. The numerical investigation shows the strong dependence of $\nu_{\max}$ frequency with respect to quantum numbers. We compare our results with recent constraints from the Laser Interferometer Gravitational-wave Observatory (LIGO) and find that it is possible to test the quantum effects in the early Universe.


## 1. Introduction

Loop Quantum Gravity (LQG) introduces strong modifications to the standard description of the early Universe. The main difference with the classical approach is the avoidance of an initial singularity [1]. This effect leads to the bouncing solution on the semi-classical level [2,3]. Another interesting property is the occurrence of the super-inflationary phase induced by quantum effects [4]. This phase is in fact not long enough to explain the observed flatness of the Universe, but after this phase the Universe has proper initial conditions to start the standard slow-roll inflation. In this scenario a inflaton field firstly climbs up the potential hill and then stops before a slow-roll phase, producing the running of the spectral index [5]. The production of scalar perturbations during super-inflationary phase is investigated in the papers [6–8].

In this Letter we consider the transition from the quantum to classical universe through the super-inflationary phase. Because during this transition the main contribution to the energy of the inflaton field comes from the kinetic part, in calculations, we neglect the contribution from the potential energy. It is worthwhile to note here that the super-inflationary phase inducted by LQG is a generic property and does not depend on a kind of the field which fills the Universe. For analytical considerations we solve dynamical equations in the semi-classical and classical regimes and then we match them. It is done for the value of the scale factor

$$a_0 = a_* = \sqrt{\frac{\gamma j}{3}} l_{\mathrm{Pl}}, \tag{1}$$

where $j$ is a half-integer quantization parameter, $l_{\mathrm{Pl}}$ is a Planck length and $\gamma$ is the Barbero–Immirzi parameter. The latter parameter

$$\gamma = \frac{\ln 2}{\pi\sqrt{3}} \tag{2}$$

comes from calculations of black-holes entropy [9]. Below the value $a_*$ non-perturbative modifications become important. We derive the equation for tensor modes in the LQG scenario. We

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investigate numerically effects of loop corrections. Neglecting quantum corrections to equation for tensor modes and holding underlying loop dynamics we study analytically creation of gravitons. We calculate the spectrum of tensor perturbations during the super-inflationary phase and the density of gravitons produced during the transition from the semi-classical to classical universe. Such gravitons give contributions to the stochastic background of gravitational waves. Nowadays the detectors like LIGO [10] aim at the detection of these stochastic gravitational waves [11]. Usually to describe the spectrum of gravitational waves the parameter

$$\Omega_{\mathrm{gw}}(\nu)=\frac{\nu}{\rho_c}\frac{d\rho_{\mathrm{gw}}}{d\nu} \tag{3}$$

is introduced. Here $\rho_c$ is the current critical density, $\rho_{\mathrm{gw}}$ is the density of gravitational waves and $\nu$ is the physical frequency measured today. The recent LIGO constraint for this parameter is $\Omega_{\mathrm{gw}} < 6.5\times 10^{-5}$ [12]. We also calculate the value of the function $\Omega_{\mathrm{gw}}(\nu)$ in the model and compare it with LIGO constraints.

## 2. The semi-classical dynamics

Loop quantum gravity introduces strong modifications to the dynamical equations in the semi-classical regime. These modifications come from the expression for the density operator [13]

$$d_j(a)=D(q)\frac{1}{a^3}, \tag{4}$$

where $q$ is defined as follows

$$q\equiv\frac{a^2}{a_*^2} \tag{5}$$

and for the semi-classical universe ($l_{\mathrm{Pl}} < a \ll a_*$) the quantum correction factor has a form [14]

$$D(q) = q^{3/2}\left\{\frac{3}{2l}\left(\frac{1}{l+2}\left[(q+1)^{l+2}-|q-1|^{l+2}\right] - \frac{q}{1+l}\left[(q+1)^{l+1}-\mathrm{sgn}(q-1)|q-1|^{l+1}\right]\right)\right\}^{3/(2-2l)}. \tag{6}$$

Here $l$ is the ambiguous parameter of quantization constrained by $0<l<1$ [15]. The Hamiltonian for the scalar field in the flat FRW universe has a form

$$\mathcal{H}=\frac{1}{2}d_j(a)p_\phi^2+a^3V(\phi),\quad \text{where } p_\phi=d_j^{-1}(a)\dot{\phi}. \tag{7}$$

This lead to the equation of motion of the form

$$\ddot{\phi}+\left(3H-\frac{\dot{D}}{D}\right)\dot{\phi}+D\frac{dV}{d\phi}=0. \tag{8}$$

The Friedmann and Raychaudhuri equations for the universe filled with a scalar field are respectively

$$H^2=\frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2D}+V(\phi)\right], \tag{9}$$

$$\frac{\ddot{a}}{a}=-\frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{D}\left(1-\frac{\dot{D}}{4HD}\right)-V(\phi)\right]. \tag{10}$$

From Eqs. (8) and (9) we obtain the relation

$$\dot{H}=-4\pi G\frac{\dot{\phi}^2}{D}\left(1-\frac{\dot{D}}{6HD}\right). \tag{11}$$

Due to quantum correction $D$ in the region ($l_{\mathrm{Pl}} < a \ll a_*$), the expression in the bracket can be negative, leading to $\dot{H}>0$ (super-inflation). If $a\gg a_*$ then $D\approx 1$ leading to $\dot{H}<0$ (deceleration). For $a\ll a_*$ the approximation of expression (6) has a form

$$D(q)\approx\left(\frac{3}{1+l}\right)^{3/(2-2l)}\left(\frac{a}{a_*}\right)^{3(2-l)/(1-l)}. \tag{12}$$

We use this approximation to calculate the dynamics in the semi-classical region. Now

$$\frac{\dot{D}}{HD}=\frac{3(2-l)}{1-l}>6 \tag{13}$$

leading to the phase of acceleration, see Eq. (11). Putting (13) into Eq. (11) and combining with (9) we obtain the equation for the scale factor

$$aa''-(a')^2\left[2+\frac{3}{2}\frac{l}{1-l}\right]=0, \tag{14}$$

where prime means the derivative in respect to the conformal time $d\tau=dt/a$. We assume here $V(\phi)=0$ as it was mentioned in Section 1. The solution of (14) is of the form

$$a\propto(-\tau)^{-2\frac{1-l}{2+l}}. \tag{15}$$

To calculate the solution in the classical regime we take $D=1$. In this case the equation for the scale factor has a form

$$aa''+(a')^2=0. \tag{16}$$

Now we match two solutions, from two regions, at some $\tau_0$ as follows

$$a_1(-\tau_0)=a_2(-\tau_0), \tag{17}$$

$$a_1'(-\tau_0)=a_2'(-\tau_0). \tag{18}$$

Where region 2 is classical and region 1 is semi-classical. The value of the chosen conformal time $\tau_0$ corresponds to the scale factor $a_*$. After matching we obtain the solution of the form

$$a_1(\tau)=a_*\left(-\frac{\tau}{\tau_0}\right)^{-2\frac{1-l}{2+l}}\quad\text{for } \tau<-\tau_0, \tag{19}$$

$$a_2(\tau)=a_*\sqrt{4\frac{1-l}{2+l}\left(\frac{\tau_0+\tau}{\tau_0}\right)+1}\quad\text{for } \tau>-\tau_0. \tag{20}$$

This solution is shown in Fig. 1 together with the numerical solution. The upper curve corresponds to the evolution of the scale factor $a(\tau)$, while the bottom curve does to the first derivative of the scale factor in respect to the conformal time. The obtained solution is of course only an approximation of the real evolution, however it is sufficiently exact to be used in analytical calculations. We find the agreement with the numerically calculated evolution of the scale factor and the Hubble rate obtained by Tsujikawa et al. [5]. In the future investigations we use both numerical and approximate analytical solutions.

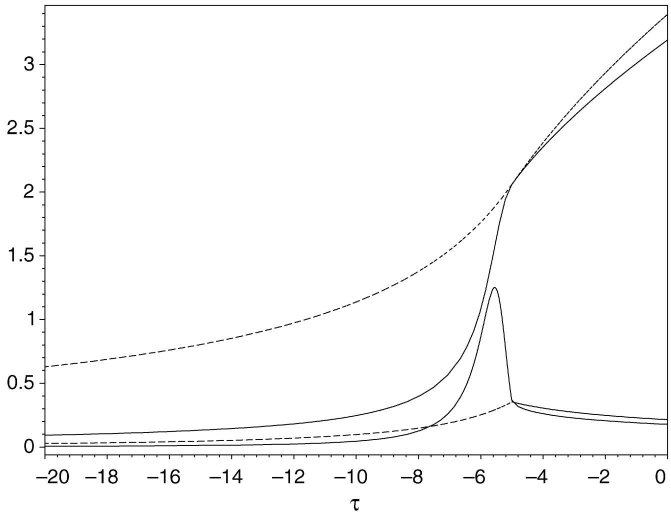


Fig. 1. The evolution of the scale factor $a$ (upper curve) and $a'$ (lower curve) in the conformal time (with $\tau_0 = 5$, $j = 100$ and $l = 0.1$). The dashed line presents the approximate solution and the solid line corresponds to the numerical solution. The initial conditions are fixed for $a_*$.

## 3. Evolution of tensor modes in loop quantum cosmology

Tensor perturbations $h_{ij}$ to the FRW metric we can express as

$$ds^2 = a^2(\tau)\big[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i\, dx^j\big], \tag{21}$$

with $|h_{ij}| \ll 1$. Using constraints $h^i_i = \nabla_i h^i_j = 0$ we can see that tensors $h_{ij}$ have only two independent components $h^1_1 = -h^2_2 = h_+$ and $h^2_1 = h^1_2 = h_\times$. These components correspond to two polarizations of gravitational waves. Since tensor modes of perturbation are not coupled to the scalar field source, we can obtain equations for them from the variation of the action

$$\begin{aligned} S^{(2)}_t &= \frac{1}{64\pi G}\int d^4x\, a^3\left[\partial_t h^i_j \partial_t h^j_i - \frac{1}{a^2}\nabla_k h^i_j \nabla_k h^j_i\right] \\ &= \frac{1}{32\pi G}\int d^4x\, a^3\left[\dot{h}^2_\times + \dot{h}^2_+ - \frac{1}{a^2}(\vec{\nabla} h_\times)^2 \right. \\ &\quad \left. - \frac{1}{a^2}(\vec{\nabla} h_+)^2\right]. \end{aligned} \tag{22}$$

For the detailed discussion of this issue see Refs. [16,17]. Both polarizations of gravitational waves are not coupled and can be treated separately. Introducing

$$h = \frac{h_+}{\sqrt{16\pi G}} = \frac{h_\times}{\sqrt{16\pi G}} \tag{23}$$

we can rewrite the action for tensor modes in the form

$$S^{(2)}_t = \frac{1}{2}\int d^4x\, a^3\left[\dot{h}^2 - \frac{1}{a^2}(\vec{\nabla} h)^2\right]. \tag{24}$$

Inverse volume corrections can be introduced now in the same way as in the scalar field case [13,14], leading to the equation of motion

$$\ddot{h} + \left(3H - \frac{\dot{D}}{D}\right)\dot{h} - D\frac{\nabla^2 h}{a^2} = 0. \tag{25}$$

In general one expects more corrections of different forms [18], but this Letter focuses on this one particular effect. In particular holonomy corrections effectively contribute a mass term in the dispersion relation for the gravitons [18]. Quantum corrections in case of vector and scalar modes was studied respectively in Refs. [19,20].

Introducing the new variable $\mu = ah$ and changing the time for conformal time we can rewrite Eq. (25) to the form

$$\mu'' - \frac{D'}{D}\mu' + \left[-D\nabla^2 - \frac{a''}{a} + \frac{a'}{a}\frac{D'}{D}\right]\mu = 0. \tag{26}$$

Since the fluctuations considered have the quantum origin we must change the classical $\mu$ for the corresponding operator $\hat{\mu}$. The field $\hat{\mu}$ and conjugate momenta $\hat{\pi}$ can be decomposed for the Fourier modes according to

$$\hat{\mu}(\vec{x},\tau) = \frac{1}{2(2\pi)^{3/2}}\int d^3k\, \big\{\hat{\mu}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \hat{\mu}^\dagger_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\big\}, \tag{27}$$

$$\hat{\pi}(\vec{x},\tau) = \frac{1}{2(2\pi)^{3/2}}\int d^3k\, \big\{\hat{\pi}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \hat{\pi}^\dagger_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\big\}, \tag{28}$$

where the relation of commutation $[\hat{\mu}(\vec{x},\tau), \hat{\pi}(\vec{x},\tau)] = i\delta^{(3)}(\vec{x} - \vec{y})$ is fulfilled. The equation for the Fourier modes is now

$$\hat{\mu}''_{\vec{k}} - \frac{D'}{D}\hat{\mu}'_{\vec{k}} + D\big[k^2 - M^2\big]\hat{\mu}_{\vec{k}} = 0, \tag{29}$$

where

$$M^2 = \frac{1}{D}\left(\frac{a''}{a} - \frac{a'}{a}\frac{D'}{D}\right) \tag{30}$$

is called the *pump field*. In the classical limit ($D = 1$) Eq. (29) assumes the known form

$$\hat{\mu}''_{\vec{k}} + \left[k^2 - \frac{a''}{a}\right]\hat{\mu}_{\vec{k}} = 0. \tag{31}$$

Because it is impossible to solve Eq. (29) analytically we must investigate the effect of quantum corrections numerically. Because $D$ is always positive we can have amplifications of the tensor modes when $k^2 < M^2$. The *pump field* function was shown in Fig. 2. In the first panel (top left) we draw *pump field* $M^2$ with neglected quantum corrections calculated numerically and with the use of solution (20). What we see is that the numerically calculated *pump field* extends the region of superadiabatic amplifications. In the next panel (top right) we compare the numerically calculated *pump field* with and without quantum corrections. We see that quantum corrections lower the region of amplification. The obtained value is however still larger than this obtained using the approximated analytical solution (20).

The friction term $D'/D$ in Eq. (29) can be positive or negative leading to the amplification or to the damping. We see that the friction term $D'/D$ is for the most of time positive leading to amplification and drastically peak to negative values in the neighbourhood of $a_*$. In fact this dependence strongly depends of a quantum number $l$. The increasing value of $l$ the negative pick goes toward to positive values and becomes less sharp.

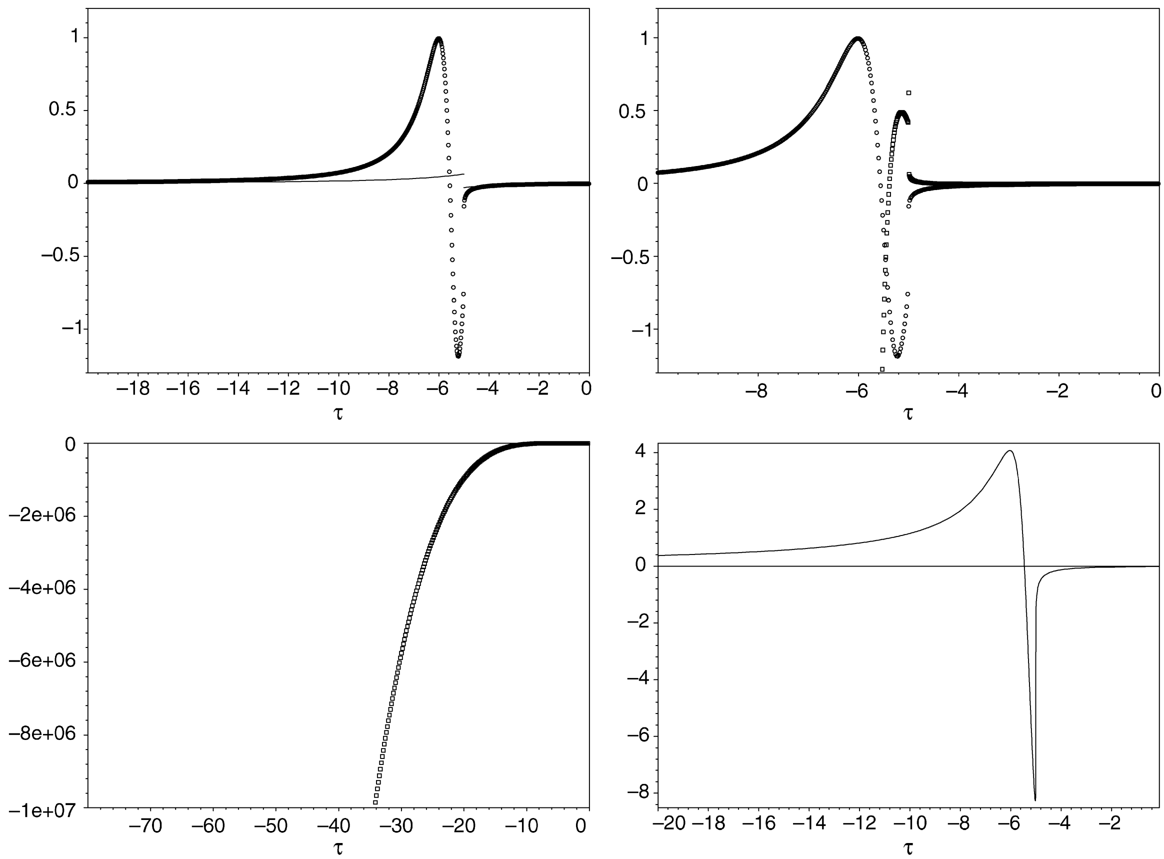


Fig. 2. Top left: *pump field* $M^2$ with neglected quantum corrections to the equation for tensor modes calculated numerically (circles) and with use of solutions (20) (line). Top right: *pump field* $M^2$ with quantum corrections to the equation for tensor modes (boxes) and without corrections (circles). Bottom left: Global behaviour of the *pump field* $M^2$ with quantum corrections to equation for tensor modes. Bottom right: Evolution of the friction term $D'/D$ in the equation for tensor modes. In all panels it is assumed $\tau_0 = 5$, $j = 100$ and $l = 0.1$.

We can now use the approximation $D = 1$ in Eq. (29) to calculate the spectrum of tensor perturbations during the super-inflationary phase. The spectrum of tensor perturbations can be now expressed using correlation function

$$\langle 0|\hat{h}^i_j(\vec{x},\tau)\hat{h}^j_i(\vec{y},\tau)|0\rangle = \frac{64\pi G}{a^2}\int \frac{d^3k}{(2\pi)^3}\left|\hat{\mu}_{\vec{k}}(\tau)\right|^2 e^{-i\vec{k}\cdot\vec{r}} \equiv \int \frac{dk}{k}\mathcal{P}_T(k)\frac{\sin kr}{kr}, \tag{32}$$

where an Einstein convention of summation was used on the left side. For considered super-inflationary phase, using expression (19), we have solution

$$\mu_k = \frac{\mathcal{N}}{\sqrt{2k}}\sqrt{-k\tau}\,H^{(1)}_{\beta+\frac{1}{2}}(-k\tau), \tag{33}$$

where

$$\mathcal{N} = \sqrt{\frac{\pi}{2}}e^{i\pi(\nu+1/2)/2} \quad \text{and} \quad \beta = 2\frac{1-l}{2+l}. \tag{34}$$

Normalization is found by correspondence to well normalized plane wave $e^{-ik\tau}/\sqrt{2k}$ for high energetical modes $|k\tau| \gg 1$. Since for us interesting are super-horizontal modes we can use approximation

$$H^{(1)}_\nu(-k\tau) \simeq -\frac{i}{\pi}\Gamma(\nu)\left(-\frac{k\tau}{2}\right)^{-\nu}. \tag{35}$$

Super-horizontal modes are these which firstly cross out the horizon and then evolve "frozen" in super-horizontal scales. Finally, in further epochs on universe, such a modes reenter horizon. These modes bring an information from earliest stages of the Universe. Given modes cross the horizon when

$$k \simeq aH = \frac{\beta}{\tau_0}\left(\frac{a}{a_*}\right)^{\frac{1}{\beta}}, \tag{36}$$

where we used definition of Hubble radius and equation of evolution (19). Finally with use of definition (32) the spectrum at horizon crossing has a form

$$\mathcal{P}_T(k) = \mathcal{A}_T^2 k^{n_T}, \tag{37}$$

where spectral index is equal

$$n_T = \frac{d\ln\mathcal{P}_T}{d\ln k} = \frac{6l}{2+l} \tag{38}$$

and normalization constant is expressed as

$$\mathcal{A}_T^2 = \frac{\Gamma^2(\beta+1/2)2^{2\beta+4}}{m_{\rm Pl}^2\pi^2 a_*^2}\left(\frac{\beta}{\tau_0}\right)^{2\beta}. \tag{39}$$

In this case tensor spectral index (38) is positive and $n_T \in (0,3)$. To compare, from the standard slow-roll inflation tensor spectral index is $n_T = -2\epsilon + \mathcal{O}(\epsilon^2)$. Unfortunately available data from CMB or from large scale structures observations are not sufficiently precise to determinate value of the tensor spectral index.

## 4. Relic gravitons from the quantum to classical universe transition

In the previous section we derived and investigated equation for tensor modes (29). Using approximations we also calculated the spectrum of gravitons which cross the horizon during the super-inflationary phase. Now we want to calculate a number of gravitons which are produced during the transition from the quantum to classical regime of evolution. Before we start it, let us calculate the width of the band of produced gravitons. It can be directly taken from condition $k^2 < M^2$. We use the classical approximation for the evolution of tensor modes for it preserves the physical picture of the graviton creation process as the previous numerical investigations indicate. In this approximation ($D = 1$) we have a maximum of frequency for $\tau_0$, so with the use of Eq. (19) we have

$$k_{\max} = \sqrt{\beta(\beta+1)}\frac{1}{\tau_0}. \tag{40}$$

In fact, as it can be seen in Fig. 2, this frequency is generally higher. The corresponding maximal frequency for the present epoch is

$$\nu_{\max} = \frac{k_{\max}}{2\pi a_*}\left(\frac{a_*}{a_{\text{today}}}\right) = \frac{\sqrt{\beta(\beta+1)}}{2\pi a_*}\frac{1}{\tau_0}\left(\frac{a_*}{a_{\text{today}}}\right). \tag{41}$$

To estimate this value we can approximate

$$\frac{a_{\text{today}}}{a_*} \sim \frac{T_{\text{Pl}}}{T_{\text{CMB}}} = \frac{1.4\times 10^{32}\ \text{K}}{3.7\ \text{K}} \simeq 10^{32}, \tag{42}$$

where $T_{\text{Pl}}$ is the Planck temperature. The other way to estimate value $a_{\text{today}}/a_*$ is to use the Friedmann equation with radiation. This gives equation

$$\frac{a_*}{a_{\text{today}}} = \sqrt{\frac{H_{\text{today}}}{H_*}} = \sqrt{\frac{H_{\text{today}} t_{\text{Pl}} \tau_0}{\beta}\sqrt{\frac{\gamma j}{3}}}, \tag{43}$$

where we use solution (20) to calculate $H_*$. To obtain a numerical value we must know $\tau_0$. We use here the constraint for energy in the form $|\dot{\phi}_i|/m_{\text{Pl}}^2 < 1$ [21] (the kinetic energy dominates over the contribution from the potential part as we mentioned before) for $a_i = \sqrt{\gamma} l_{\text{Pl}}$. Below the value of chosen $a_i$ the space becomes discrete and the smooth dynamical equations cannot be used. The boundary for the kinetic energy is introduced to prevent energies beyond the Planck scale being produced. With use of this boundary conditions and the Friedmann equation (9) with solution (20) we obtain the constraint for the conformal time $\tau_0$

$$\tau_0 > \frac{1-l}{2+l}\sqrt{\frac{3}{\pi\gamma}}\left(\frac{3}{1+l}\right)^{\frac{3}{2}\frac{1}{2-2l}}\left(\frac{3}{j}\right)^{\frac{1}{2}\frac{4-l}{1-l}}. \tag{44}$$

As an example for the model with $l = 0.1$ and $j = 100$ we obtain $\tau_0 > 0.0014$ and for the model with $l = 3/4$ and $j = 100$ we obtain $\tau_0 > 1.6 \times 10^{-8}$. We see that this boundary depends very strongly on the quantum numbers. Combining Eq. (41) with (43) we see that $\nu_{\max} \propto \tau_0^{-1/2}$, so the boundary (44) gives us also the upper constraint for a maximal value of frequency $\nu_{\max}$. For the model with $l = 0.1$ and $j = 100$ we have $\nu_{\max} < 6.6 \times 10^{14}$ Hz and for the model with $l = 3/4$ and $j = 100$ we obtain $\nu_{\max} < 2.8 \times 10^{24}$ Hz. Generally values of $\nu_{\max}$ can be smaller than boundary values. For the further studies we choose the model with $\tau_0 = 0.1$. So in this case the width of the band of relic gravitons considered is nowadays $[0, 10^3\ \text{GHz}]$ for $l = 0.1$.

Fourier modes of (27) and (28) for the super-inflationary evolution (19) can be written with the use of annihilation and creation operators as follows

$$\hat{\mu}_{\vec{k}}(\tau) = \hat{a}_{\vec{k}} f_1(k,\tau) + \hat{a}^\dagger_{-\vec{k}} f_1^*(k,\tau) \quad \text{for } \tau < -\tau_0, \tag{45}$$

$$\hat{\pi}_{\vec{k}}(\tau) = \hat{a}_{\vec{k}} g_1(k,\tau) + \hat{a}^\dagger_{-\vec{k}} g_1^*(k,\tau) \quad \text{for } \tau < -\tau_0. \tag{46}$$

In this case the values of coefficients are

$$f_1(k,\tau) = \frac{\mathcal{N}_1}{\sqrt{2k}}\sqrt{-k\tau}H_\nu^{(1)}(-k\tau), \tag{47}$$

$$g_1(k,\tau) = -\mathcal{N}_1\sqrt{\frac{k}{2}}\sqrt{-k\tau}\left[-H_{\nu+1}^{(1)}(-k\tau) + \frac{1+2\nu}{2(-k\tau)}H_\nu^{(1)}(-k\tau)\right], \tag{48}$$

where

$$\mathcal{N}_1 = \sqrt{\frac{\pi}{2}}e^{i\pi(\nu+1/2)/2} \quad \text{and} \quad \nu = \beta + \frac{1}{2}. \tag{49}$$

Similarly, modes of (27) and (28) for the classical evolution (20) we can be written down as

$$\hat{\mu}_{\vec{k}}(\tau) = \hat{b}_{\vec{k}} f_2(k,\tau) + \hat{b}^\dagger_{-\vec{k}} f_2^*(k,\tau) \quad \text{for } \tau > -\tau_0, \tag{50}$$

$$\hat{\pi}_{\vec{k}}(\tau) = \hat{b}_{\vec{k}} g_2(k,\tau) + \hat{b}^\dagger_{-\vec{k}} g_2^*(k,\tau) \quad \text{for } \tau > -\tau_0. \tag{51}$$

Where the coefficients of decomposition are

$$f_2(k,\tau) = \mathcal{N}_2\sqrt{1+4\frac{1-l}{2+l}\left(\frac{\tau_0+\tau}{\tau_0}\right)}H_0^{(2)} \times (k\tau + k\zeta)\exp(ik\zeta), \tag{52}$$

$$g_2(k,\tau) = \frac{\mathcal{N}_2}{\tau_0}\left[\frac{H_0^{(2)}(k\tau+k\zeta)}{\sqrt{1+4\frac{1-l}{2+l}(\frac{\tau_0+\tau}{\tau_0})}}\frac{2(1-l)}{2+l} - k\tau_0\sqrt{1+4\frac{1-l}{2+l}\left(\frac{\tau_0+\tau}{\tau_0}\right)}H_1^{(2)}(k\tau+k\zeta)\right]\exp(ik\zeta), \tag{53}$$

with

$$\mathcal{N}_2 = \frac{\sqrt{\pi}}{4}\sqrt{\tau_0}\sqrt{\frac{2+l}{1-l}}e^{-i\pi/4} \quad \text{and} \quad \zeta = \tau_0\frac{3}{4}\frac{2-l}{1-l}, \tag{54}$$

where $H^{(2)}$ is the Haenkel function of the second kind.

The main idea of particles creation during transition comes from the Bogoliubov transformation

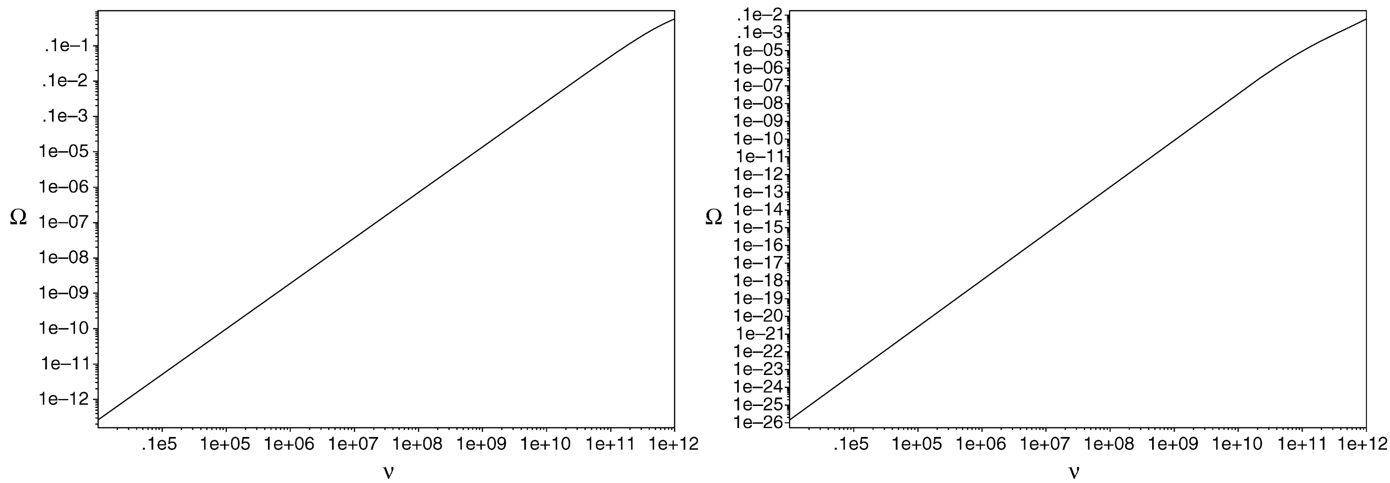


Fig. 3. Left: The function $\Omega_{gw}(\nu)$ with $j = 100$, $\tau_0 = 0.1$ and $l = 0.1$. Right: The function $\Omega_{gw}(\nu)$ with $j = 100$, $\tau_0 = 0.1$ and $l = 3/4$. The frequency scales in Hertz.

$$\hat{b}_{\vec{k}} = B_+(k)\hat{a}_{\vec{k}} + B_-(k)^*\hat{a}^\dagger_{-\vec{k}}, \tag{55}$$

$$\hat{b}^\dagger_{\vec{k}} = B_+(k)^*\hat{a}^\dagger_{\vec{k}} + B_-(k)\hat{a}_{-\vec{k}}, \tag{56}$$

where from relations of commutation $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = \delta^{(3)}(\vec{k}-\vec{p})$ and $[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{p}}] = \delta^{(3)}(\vec{k}-\vec{p})$ we have $|B_+|^2 - |B_-|^2 = 1$. In the quantum phase we have $\hat{a}_{\vec{k}}|0_{\text{in}}\rangle = 0$ where $|0_{\text{in}}\rangle$ is the vacuum state of this phase. In the final classical epoch, similarly $\hat{b}_{\vec{k}}|0_{\text{out}}\rangle = 0$ what defines the new vacuum state $|0_{\text{out}}\rangle$. But since we are in the Heisenberg description the true vacuum state in the classical phase is $|0_{\text{in}}\rangle$ and thanks to the mixing from the Bogoliubov transformation (55) we have $\hat{b}_{\vec{k}}|0_{\text{in}}\rangle = B_-(k)^*\hat{a}^\dagger_{-\vec{k}}|0_{\text{in}}\rangle$. So when $B_-(k)$ is the nonzero coefficient we have the production of particles (gravitons) in the final state. What we need now is to calculate coefficients of the Bogoliubov transformation $B_-(k)$ and $B_+(k)$ which can be written as

$$B_-(k) = \frac{f_1(-\tau_0)g_2(-\tau_0) - g_1(-\tau_0)f_2(-\tau_0)}{f_2^*(-\tau_0)g_2(-\tau_0) - g_2^*(-\tau_0)f_2(-\tau_0)}, \tag{57}$$

$$B_+(k) = \frac{f_1(-\tau_0)g_2^*(-\tau_0) - g_1(-\tau_0)f_2^*(-\tau_0)}{f_2(-\tau_0)g_2^*(-\tau_0) - g_2(-\tau_0)f_2^*(-\tau_0)}. \tag{58}$$

Since the total momentum of produced gravitons is conserved we can write the expression for the number of produced particles

$$\bar{n}_{\vec{k}} = \frac{1}{2}\langle 0_{\text{in}}|\left[\hat{b}^\dagger_{\vec{k}}\hat{b}_{\vec{k}} + \hat{b}^\dagger_{-\vec{k}}\hat{b}_{-\vec{k}}\right]|0_{\text{in}}\rangle = \left|B_-(k)\right|^2. \tag{59}$$

As we can see, to calculate a number of gravitons we only need to know the coefficient $B_-(k)$. Now we can calculate the function $\Omega_{gw}(\nu)$ defined in Eq. (3). The essential energy density is from the relation

$$d\rho_{gw} = 2 \cdot \hbar\omega \cdot \frac{4\pi\omega^2\, d\omega}{(2\pi c)^3} \cdot \bar{n}_{\vec{k}}, \tag{60}$$

where factor 2 comes from two polarizations of gravitational waves. With the use of relation (41) we finally obtain the equation

$$\Omega_{gw}(\nu) = 3.7 \cdot 10^{-49} h_0^{-2} \nu^4 \bar{n}\left(\sqrt{\beta(\beta+1)}\frac{\nu}{\nu_{\max}}\right), \tag{61}$$

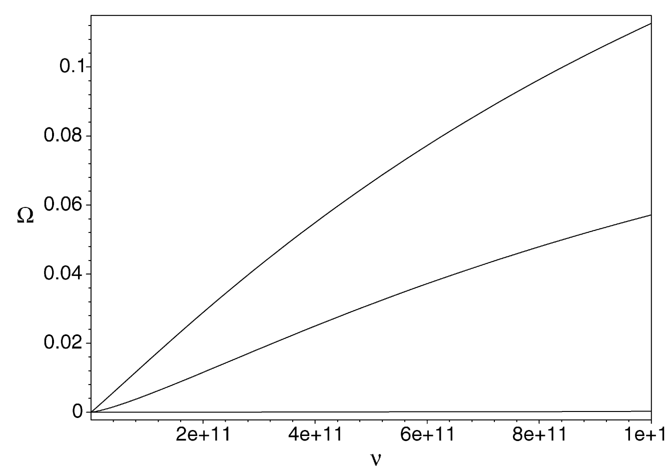


Fig. 4. The function $\Omega_{gw}(\nu)$ for $l = 0.01, 0.1, 3/4$ (from top to bottom), $\tau_0 = 0.1$ and $j = 100$. The frequency scale in Hertz.

where $h_0$ is the normalized Hubble rate $h_0 = H_0/100\ \text{km}^{-1}\,\text{s}\,\text{Mpc}$. We compute this function and show it in the logarithmic plot with $l = 0.1$ and $l = 3/4$ (Fig. 3). This spectrum is characterized by a hard branch with the maximum for $\sim 10^{12}$ Hz for $l = 0.1$ and $\sim 10^{11}$ Hz for $l = 3/4$. In this limit $\Omega_{gw}$ approaches respectively to $\sim 10^{-1}$ and $\sim 10^{-5}$.

When the high energy region is shown only, the dependence $\Omega_{gw}(\nu)$ on the quantum parameter $l = 0.01, 0.1, 3/4$ is exhibited (Fig. 4).

As we mentioned in Section 1, recent constraints from LIGO are $\Omega_{gw} < 6.5 \times 10^{-5}$ [12]. The LIGO observations are however concentrated in the region of $\sim 10^2$ Hz. From Loop Quantum Cosmology we have in this region $\Omega_{gw} \sim 10^{-14}$ (for $l = 0.1$), what is extremely below the observational sensitivity. The numerical values obtained by us contain estimations of the time of transition to classical universe. The used value should be somehow proper to the order of magnitude. So we expect also similar deviations of $\Omega_{gw}(\nu)$.

The spectrum obtained here is not a distinct feature of loop quantum cosmology. As it was shown by Giovannini [22] a similar high energy branch was obtained in the quintessential inflationary model. Calculations based on string cosmology lead also to similar results [23]. To compare, for the standard inflationary models the spectrum is flat.

## 5. Summary

Loop quantum cosmology has received much attention in the theoretical astrophysics. But what was lacked so far was empirical consideration of this theory. Bojowald indicated the quantum effects are negligible small at the present epoch but they can potentially tested [24]. Along Bojowald's lines we showed that gravitational waves can be the real observable for testing loop quantum gravity effects.

We have considered the transition from the semi-classical to classical universe described by loop quantum cosmology. In the analytical approximation we obtained the tensor energy spectrum of the relic gravitons from the super-inflationary phase. The analytical model takes into consideration the corrections to dynamical evolution only. While taking corrections to the equation for the tensor modes this equation cannot be solved analytically, so it is only possible to consider it numerically. The numerical investigation of the equation for tensor modes gave us that lower $\nu_{\max}$ is admissible when the loop quantum effects are incorporated. The loop quantum gravity effects product additional damping during the production of gravitons. This is a challenge for future investigation—the full numerical analysis of this model.

When we considered the production of gravitons during the transition phase the spectrum of these gravitons is characterized by the hard branch. The corresponding value of the parameter $\Omega_{\text{gw}}$, in its maximum, is $\Omega_{\text{gw}} \sim 10^{-7} \ldots 10^{-1}$, depending on the value of the parameter of quantization $l$. This value seems to be high in comparison with results obtained in the other models [22]. We must however keep in mind that we have obtained this result neglecting corrections to the equation for the tensor modes. The inverse volume correction and other corrections expected in the loop quantum cosmology can potentially strongly influence for the obtained values. In particular the inverse volume corrections produce damping of the tensor modes what lower the value of the parameter $\Omega_{\text{gw}}$.

In the region of the LIGO highest sensitivity we obtained the very small value of the parameter $\Omega_{\text{gw}}$, namely $\sim 10^{-14}$ for $l = 0.1$ and $\sim 10^{-28}$ for $l = 3/4$. As we mentioned, the similar hard branch is also a feature of quintessential inflationary and string cosmology models. This work gives the motivation to the further theoretical investigations and search for high energetic gravitational waves.

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