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Effective dynamics of the closed loop quantum cosmology

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Abstract. In this paper we study dynamics of the closed FRW model with holonomy corrections coming from loop quantum cosmology. We consider models with a scalar field and cosmological constant. In case of the models with cosmological constant and free scalar field, dynamics reduce to 2D system and analysis of solutions simplify. If only free scalar field is included then universe undergoes non-singular oscillations. For the model with cosmological constant, different behaviours are obtained depending on the value of Λ. If the value of Λ is sufficiently small, bouncing solutions with asymptotic de Sitter stages are obtained. However if the value of Λ exceeds critical value $\Lambda_c = \frac{\sqrt{3}m_{\text{Pl}}^2}{2\pi\gamma^3} \simeq 21 m_{\text{Pl}}^2$ then solutions become oscillatory. Subsequently we study models with a massive scalar field. We find that this model possess generic inflationary attractors. In particular field, initially situated in the bottom of the potential, is driven up during the phase of quantum bounce. This subsequently leads to the phase of inflation. Finally we find that, comparing with the flat case, effects of curvature do not change qualitatively dynamics close to the phase of bounce. Possible effects of inverse volume corrections are also briefly discussed.

Keywords: inflation, quantum cosmology

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Contents

1 Introduction

In the recent years methods of the Loop Quantum Gravity (LQG) [1] have been successfully applied to quantise cosmological minisuperspace models. General prediction of this approach, called Loop Quantum Cosmology (LQC) [2], is avoidance of the initial singularity. Namely in LQC singularity is replaced by the phase of bounce. This result seems to be generic for the homogeneous cosmologies and was confirmed in the numerous investigations. Loop quantisation of the homogeneous and isotropic FRW models have been performed in ref. [3–5] ($K = 0$), ref. [6, 7] ($K = 1$) and ref. [8, 9] ($K = -1$). In the papers cited above strict, fully quantum considerations were performed. However models only with a free scalar field and cosmological constant were studied.

Dynamics of the models also can be traced with the effective equations of motion. Then classical dynamics is modified with certain nonperturbative quantum gravitational corrections. These corrections becomes unimportant for the low energy densities leading to the classical equations of motion. However for the scales comparable with the Planck energy densities corrections produce effective "quantum repulsion" leading to the bounce. Studies of the effective dynamics have recently attracted substantial interest. It is mainly due to the fact that it gives the access to studying various physical consequences of the loop quantisation. Moreover, semi-classical considerations are in many points simpler than the fully quantum treatment. Often approximate solution can be found before the strict quantum results are obtained. In particular models with self-interacting scalar field can be easily studied in this effective formulation. Therefore both fully quantum and effective methods are complementary to each other. Since now effective flat FRW [10, 11] and Bianchi I [12] models in LQC were studied. In the present paper we study effective dynamics of the closed ($K = 1$) FRW model.

We have to stress that alternative effective description is developed by Bojowald and his collaborators. In the Bojowald's approach [13], information about dispersions and higher moments of the observables can also be obtained. Recently this approach has been applied to Bianchi I model [14]. However we do not apply this approach in our considerations.

Recently also new approach to quantise loop quantum cosmological models has been proposed [15, 16]. This formulation leads to the bouncing solutions but no effects of so called Thiemann's term, which leads to inverse volume corrections, are present. Obtained effective dynamics is therefore the same like this studied in [10, 11] where only holonomy corrections were taken into account.

Canonical variables for the closed FRW model are similar like in the flat case. Namely original Ashtekar variables (A^i_a, E^a_i) are parametrised by the conjugated (c, p) variables with the Poisson bracket

$$\{c, p\} = \frac{8\pi G\gamma}{3}. \tag{1.1}$$

Parameters (c, p) can be expressed in terms of the standard FRW variables in the following way

$$p = a^2 \left(\frac{l_0}{a_0}\right)^2 \quad \text{and} \quad c = (\gamma\dot{a} + 1)\frac{l_0}{a_0} \tag{1.2}$$

where $a_0 = 2$, fiducial volume $V_0 = 2\pi^2 a_0^3 = 16\pi^2$ and $l_0 = V_0^{1/3} = (16\pi^2)^{1/3} = (2\pi^2)^{1/3}a_0$. It will become useful when correspondence with classical case will be studied.

In this paper value of Barbero-Immirzi parameter $\gamma = 0.2375$, following ref. [17], is assumed. We also use notation $\kappa = 8\pi G$.

2 Effective equations for the closed FRW cosmology

In this section we derive equations of motion for the closed FRW model with holonomy corrections. Effects of holonomies are introduced here by the modifications in the classical Hamiltonian. Form of these modifications was derived in ref. [6] for the model with a free scalar field only. This was obtained from the purely quantum considerations of the closed FRW model in the framework of LQC. It was also shown that solutions of the modified classical equations trace the mean values of the quantum operators, obtained in the fully quantum treatment. This was however shown for the free field case only. In this paper we apply the obtained in the free field case holonomy corrections to the model with self-interacting scalar field. We do not take into account the so-called inverse volume corrections.

The effects of holonomy corrections are introduced by replacement of the curvature of the Ashtekar connection F^k_{ab} by its discretised version, where the area scale of discretisation is Δ. Therefore we could expect that only gravity part containing the F^k_{ab} term is modified. In such a case the matter part should hold the classical form. Moreover the effect of self-interacting scalar field does not influence on the gravity part of the Hamiltonian in this approach. In this paper we follow this point of view. However it is not only possible way of introducing quantum gravity effects to the classical Hamiltonian. The Bojowald's approach [13] gives systematic method of deriving the effective equations of motion. In this approach the effect of self-interacting scalar field influences significantly the form of the effective equations. In this paper we present alternative approach of deriving the effective equations of motion. Our equations are however not effective in the strict sense. Namely they not base on the systematic method of deriving effective equations of motion for the given quantum system as Bojowald do. In our approach we introduce quantum modifications directly to the classical Hamiltonian. The effectiveness of our method should be therefore understood in this sense. The modifications applied should be furthermore seen rather as phenomenological corrections. In this paper we call this approach effective, however it should not be misled with the effectiveness in the strict sense represented in the Bojowald's works. However the both methods give the same results in case of the free scalar field. In case of the self-interacting field we expect different predictions from the both methods.

Starting point of our considerations is the effective Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{-3\sqrt{p}N}{8\pi G\gamma^2\bar{\mu}^2}\left[\sin^2\left(\bar{\mu}c - \bar{\mu}\frac{l_0}{2}\right) - \sin^2\left(\bar{\mu}\frac{l_0}{2}\right) + (1+\gamma^2)\frac{\bar{\mu}^2 l_0^2}{4}\right] + \mathcal{H}_{\rm m} \tag{2.1}$$

which was derived in ref. [6]. As we have mentioned, this is classical Hamiltonian with curvature of the Ashtekar connection F^k_{ab} replaced by its discretised version. Here $\bar{\mu}$ is discretisation parameter and its form depends on the particular quantisation scheme in LQC. In this paper we choose so called $\bar{\mu}$-scheme, then

$$\bar{\mu} = \sqrt{\frac{\Delta}{p}} \tag{2.2}$$

where $\Delta = 2\sqrt{3}\pi\gamma l_{\rm Pl}^2$. Earlier, μ_0-scheme was also used, however it was realised later that it has wrong classical limit. Recently other quantisation scheme was proposed [18] and studied in ref. [19, 20]. In this scheme new types of finite scale factor singularities appears. Another possibility is those presented in [15, 16] where $\bar{\mu} = \lambda/\sqrt{p}$ and λ is some unknown constant unrelated with Δ.

As a matter content we consider homogeneous scalar field with the Hamiltonian

$$\mathcal{H}_{\rm m} = Np^{3/2}\left(\frac{p_\phi^2}{2p^3} + V(\phi)\right). \tag{2.3}$$

With this assumption it will be possible to study realisation of the inflationary phase in the bouncing universe. Energy density of the scalar field can be derived from the following expression

$$\rho := \frac{1}{p^{3/2}}\frac{\partial\mathcal{H}_{\rm m}}{\partial N}. \tag{2.4}$$

Since the Hamiltonian constraint $\frac{\partial\mathcal{H}_{\rm eff}}{\partial N} = 0$ is fulfilled we obtain

$$\sin^2\left(\bar{\mu}c - \bar{\mu}\frac{l_0}{2}\right) = \left[\sin^2\left(\bar{\mu}\frac{l_0}{2}\right) - (1+\gamma^2)\frac{\bar{\mu}^2 l_0^2}{4}\right] + \frac{\rho}{\rho_c} \tag{2.5}$$

where we have defined critical density

$$\rho_{\rm c} = \frac{3}{\kappa\gamma^2\Delta} = \frac{\sqrt{3}}{16\pi^2\gamma^3 l_{\rm Pl}^4} \simeq 0.82\rho_{\rm Pl}. \tag{2.6}$$

In the rest of the paper we choose the gauge $N = 1$.

The equations of motion can be derived with the use of Hamilton equation

$$\dot{f} = \{f, \mathcal{H}_{\rm eff}\} \tag{2.7}$$

where the Poisson bracket is defined as follows

$$\{f, g\} = \frac{8\pi G\gamma}{3}\left[\frac{\partial f}{\partial c}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial c}\right] + \left[\frac{\partial f}{\partial \phi}\frac{\partial g}{\partial p_\phi} - \frac{\partial f}{\partial p_\phi}\frac{\partial g}{\partial \phi}\right]. \tag{2.8}$$

JCAP11(2009)014

From this definition we can retrieve the elementary brackets

$$\{c, p\} = \frac{8\pi G\gamma}{3} \quad \text{and} \quad \{\phi, p_\phi\} = 1. \tag{2.9}$$

Based on eq. 2.7 we derive equation for the scalar field part

$$\dot{\phi} = \{\phi, \mathcal{H}_{\text{eff}}\} = p^{-3/2} p_\phi, \tag{2.10}$$

$$\dot{p_\phi} = \{p_\phi, \mathcal{H}_{\text{eff}}\} = -p^{3/2} \frac{dV}{d\phi} \tag{2.11}$$

what leads to the classical equation

$$\ddot{\phi} + \frac{3}{2}\frac{\dot{p}}{p}\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.12}$$

In the present paper we do not take the so-called inverse volume corrections into account. Therefore equation of motion for the scalar fields does not feel directly the quantum corrections. The correction is however indirect by the different background dynamics modified by the holonomy corrections. This dependence contributes by the friction term in the above equation.

For the gravity part we calculate

$$\dot{p} = \{p, \mathcal{H}_{\text{eff}}\} = -\frac{8\pi G\gamma}{3}\frac{\partial \mathcal{H}_{\text{eff}}}{\partial c} \tag{2.13}$$

what together with eq. 2.5 give us effective Friedmann equation

$$H^2 := \left(\frac{\dot{p}}{2p}\right)^2 = \frac{8\pi G}{3}\frac{1}{\rho_{\text{c}}}(\rho - \rho_1(p))(\rho_2(p) - \rho) \tag{2.14}$$

where

$$\rho_1(p) = -\rho_{\text{c}}\left[\sin^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) - (1+\gamma^2)\frac{\Delta l_0^2}{4p}\right] \approx \frac{3}{\kappa a^2}, \tag{2.15}$$

$$\rho_2(p) = \rho_{\text{c}}\left[1 - \sin^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) + (1+\gamma^2)\frac{\Delta l_0^2}{4p}\right] \approx \rho_{\text{c}} + \frac{3}{\kappa a^2}. \tag{2.16}$$

It is clear from the above expressions that relation $\rho_2(p) - \rho_1(p) = \rho_{\text{c}}$ is fulfilled. Approximations performed in (2.15) and (2.16) are valid for $p \gg \frac{\Delta l_0^2}{4} \simeq 18.9\, l_{\text{Pl}}^2$.

Alternatively expression for the modified Friedmann equation can be written as follows

$$H^2 = \frac{8\pi G_{\text{eff}}}{3}\rho - \frac{K_{\text{eff}}}{a^2} \tag{2.17}$$

where effective constants are expressed as follows

$$G_{\text{eff}} = G\left[1 + 2\frac{\rho_1}{\rho_{\text{c}}} - \frac{\rho}{\rho_{\text{c}}}\right], \tag{2.18}$$

$$K_{\text{eff}} = \frac{\kappa}{3}\left(\frac{a_0}{l_0}\right)^2 p\frac{\rho_1(p)\rho_2(p)}{\rho_{\text{c}}}. \tag{2.19}$$

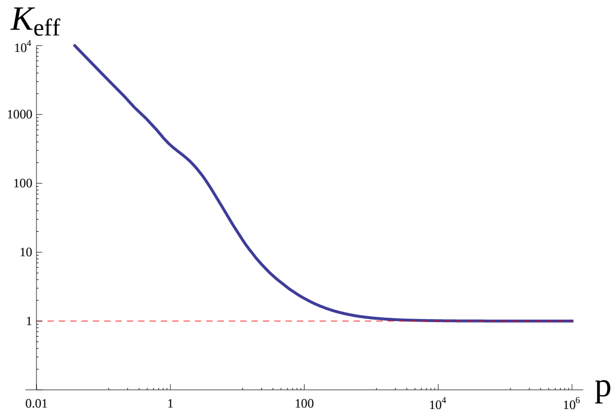


Figure 1. Effective parameter of the curvature $K_{\rm eff}$.

Expression for the $K_{\rm eff}$ does not depends on the energy density ρ. Therefore it can be studied without specifying form of the matter. We expect that classical limit $K_{\rm eff} \to 1$ for $p \to \infty$ should be recovered. It is in fact a case, what it is clear from figure 1. For $p \to 0$, factor $K_{\rm eff}$ grows and can potentially play important role. We study this issue in the subsequent sections.

Defining energy density and pressure of the scalar field

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{2.20}$$

$$\mathcal{P} = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{2.21}$$

we can derive equation

$$\frac{dH}{dt} = -4\pi G \left[(\rho + \mathcal{P}) - M(p)\rho_c\right] \left[1 + 2\frac{\rho_1}{\rho_c} - 2\frac{\rho}{\rho_c}\right] \tag{2.22}$$

where to simplify notation we have defined

$$M(p) = (1+\gamma^2)\frac{\Delta l_0^2}{6p} - \frac{l_0}{3}\sqrt{\frac{\Delta}{p}} \sin\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) \cos\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right). \tag{2.23}$$

It is worth to mention that taking $l_0 = 0$ in (2.14) and (2.22) we obtain

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right), \tag{2.24}$$

$$\frac{dH}{dt} = -4\pi G(\rho + \mathcal{P})\left[1 - 2\frac{\rho}{\rho_c}\right] \tag{2.25}$$

recovering the case $K = 0$.

3 Models with Λ and free scalar field as 2D dynamical systems

Effective loop dynamics of the flat FRW models with Λ and free scalar field was studied in [11]. In this reference, fully analytical solutions of the model were found. Unfortunately in case

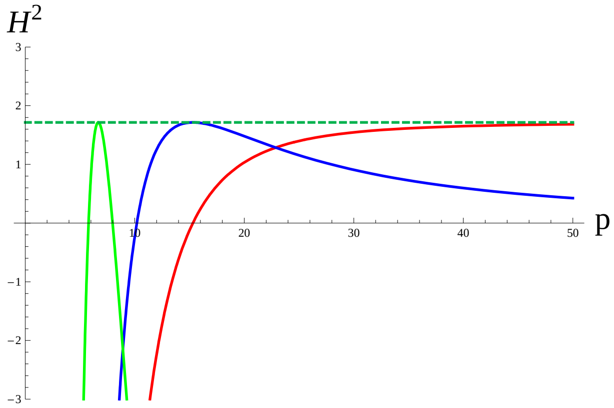


Figure 2. Square of the Hubble factor for different values of ρ_Λ. From right $\rho_\Lambda = 0.5\rho_c$ (red), $\rho_\Lambda = \rho_c$ (blue) and $\rho_\Lambda = 2.5\rho_c$ (green). Dashed line represents $H^2_{\rm max} = \frac{\kappa}{12}\rho_c$.

of the closed model the equations become transcendental and cannot be solved analytically. Therefore other approach, to investigate dynamics, have to be applied. In our considerations we use methods of the qualitative analysis of the dynamics. Useful technique applied here is compactification of the phase space. Based on this approach it will be possible to study global properties of dynamics.

In this section we consider models with cosmological constant and free scalar field. For these models dynamics reduce to 2D dynamical systems, what simplify qualitative analysis. Moreover one has to remember that physical solutions are on the surface of Hamiltonian constraint $\mathcal{H}_{\rm eff}(p,\dot{p}) \approx 0$.[1] Therefore domain allowed for the motion reduce to one dimensional subspace. It means that on 2D phase plane $(p,\dot{p})$, only one trajectory represents physical motion. Other possible trajectories does not fulfil $\mathcal{H}_{\rm eff}(p,\dot{p}) \approx 0$. It is also interesting to note that in all cases, the same maximal value of Hubble parameter $H_{\rm max}$ is reached. In fact this is the same $H_{\rm max} = \sqrt{\frac{\kappa}{12}\rho_c}$ as this obtained for the flat models.

3.1 Cosmological constant

In the first case we consider model with cosmological constant only. We will restrict our considerations to positive values of Λ only. Energy density and pressure of the cosmological constant are given by

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \quad \text{and} \quad \mathcal{P}_\Lambda = -\frac{\Lambda}{8\pi G}. \tag{3.1}$$

Dynamics of the system is governed now only by modified Friedmann equation (2.14). Since left side of this equation is $H^2 \geq 0$ on can find physical domain of motion form this condition. We plot H^2 function given by eq. (2.14) in figure 2. We find that while $\rho_\Lambda > \rho_c$ then domain for the physical motion is bounded from two sides. Therefore we expect oscillatory solutions in this regime. While $\rho_\Lambda = \rho_c$ the upper bound reach infinity. In turn, when $\rho_\Lambda < \rho_c$ upper bound disappears. It is useful to define critical value of cosmological constant

$$\Lambda_{\rm c} = \frac{\sqrt{3}m^2_{\rm Pl}}{2\pi\gamma^3} \simeq 21 m^2_{\rm Pl}, \tag{3.2}$$

[1] The dot refers to a derivative by coordinate time.

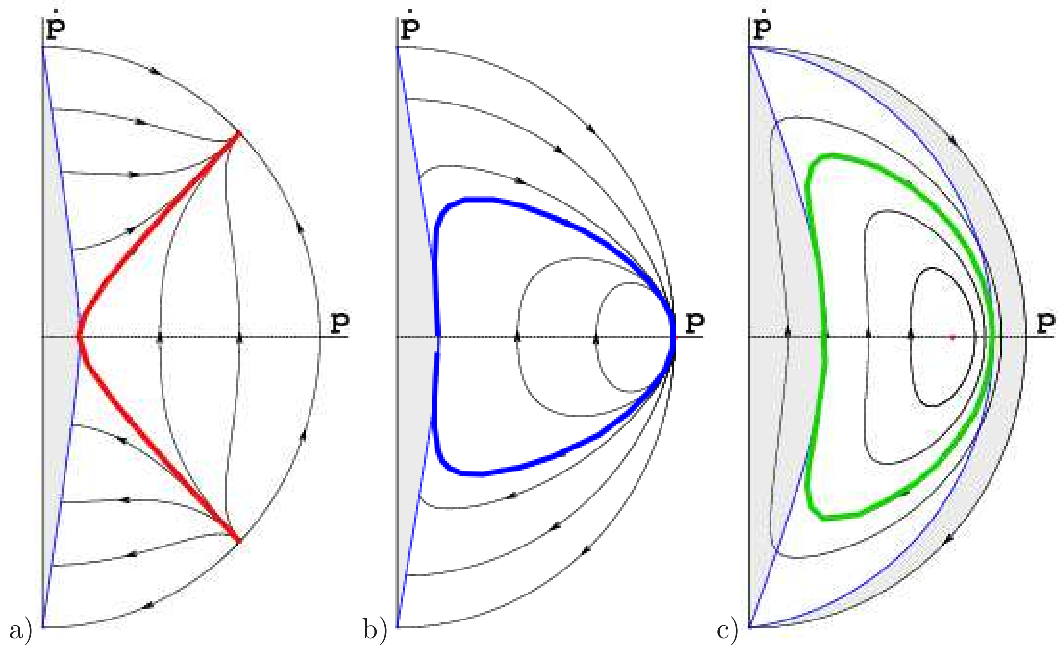


Figure 3. Phase space diagrams for the model with the cosmological constant. Shaded regions are unphysical because of $H^2 < 0$. Case a) $0 < \rho_\Lambda < \rho_c$, b) $\rho_\Lambda = \rho_c$, c) $\rho_\Lambda > \rho_c$. Physical solutions, on the surface of Hamiltonian constraint $\mathcal{H}_{\text{eff}}(p, \dot{p}) \approx 0$, are distinguished by thick (red, blue, green) lines.

because bifurcation of solutions occur when value of cosmological constant cross Λ_{c}. This critical value was introduced in ref. [11] while studying flat models in LQC. It was show there that there is no physical solutions for $\Lambda > \Lambda_{\text{c}}$. In the curved models we however expect physical motion in this region.

From the observations we know that the case $\Lambda > \Lambda_{\text{c}}$ is not realised in Nature. However we cannot treat this solution as unphysical. Namely, we cannot exclude that universe would re-collapse if the amount of the cosmological constant will extend the critical density. In this case we cannot expect any classical limit, since the energy density is comparable with the critical density during the whole evolution. The ratio ρ/ρ_{c} can be treated as a measure of the classically of the universe rather than an extensive variable as a total volume. Therefore the universe, in principle, can be in the quantum domain even for the large volumes. In seems that the case of $\Lambda > \Lambda_{\text{c}}$ represents such a situation.

Now we can investigate dynamics of these three cases on the phase phase portraits shown in figure 3. We see tree mentioned types of solutions. It is worth to stress that only thick lines represent physical trajectories. Other trajectories shown in figure 3 are not on the surface of Hamiltonian constraint $\mathcal{H}_{\text{eff}}(p, \dot{p}) \approx 0$. Based on approximations performed in (2.15) and (2.16) we find

$$p_{\text{min}} \approx \frac{3}{4}\frac{l_0^2}{\Lambda} \mapsto a_{\text{min}} \approx \sqrt{\frac{3}{\Lambda}}, \tag{3.3}$$

$$p_{\text{max}} \approx \frac{3}{4}\frac{l_0^2}{\Lambda - \Lambda_{\text{c}}} \mapsto a_{\text{max}} \approx \sqrt{\frac{3}{\Lambda - \Lambda_{\text{c}}}}. \tag{3.4}$$

Expression for p_{max} is however valid only for $\Lambda \geq \Lambda_{\text{c}}$. When $\Lambda < \Lambda_{\text{c}}$ we have $p_{\text{max}} \to \infty$. In this case asymptotic de Sitter stage is obtained as in the flat case.

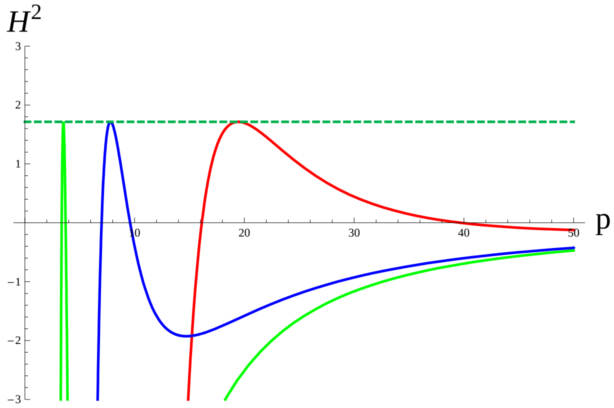


Figure 4. Square of the Hubble factor for different values of p_ϕ. From left to right $p_\phi = 20, 40, 100\, l_{\rm Pl}$. Dashed line represents $H^2_{\rm max} = \frac{\kappa}{12}\rho_c$.

3.2 Free scalar field

Now we get on to example of free scalar filed which is widely used in LQC. It is due to the fact that, in this model, scalar field can be treated as internal clock. It is of great importance when purely quantum cosmological models are constructed. However it is not necessary on the semi-classical level. Therefore models with no well defined global internal time can also be studied. Example of this will be given in the subsequent section when model with self-interacting field is studied. However even in that case well defined intrinsic time can be introduced at intervals.

Considered model with free scalar filed was studied before in ref. [6]. In this reference fully quantum analysis of the model has been done. In particular wave function for this model was computed. It was also shown that semi-classical dynamics follows precisely the quantum evolution. It is important result approving applicability of the semi-classical approach. Here we would like to complete earlier investigations.

It is worth to remind here that energy density and pressure of the free scalar field are given by

$$\rho_\phi = \frac{p_\phi^2}{2p^3} \quad \text{and} \quad \mathcal{P}_\phi = \frac{p_\phi^2}{2p^3} \tag{3.5}$$

where p_ϕ is constant of integration here. Based, as earlier, on eq. (2.14) we can find the regions $H^2 \geq 0$. This gives us regions allowed for motion. We show exemplary curves in figure 4. It can be seen that doubly restrict region is allowed for the motion. This region is enlarged with increasing value of parameter p_ϕ.

Based on analysis of positivity of H^2 function one expect that solutions are represented by oscillations. Namely universe bouncing between two edges of the region allowed for motion. It can be approved looking on phase portrait for this system, shown in figure 5. In fact physical solution is represented by closed curve. Similarly like in the previous case, physical solution lies on the surface of Hamiltonian constraint. Here we can also find

$$p_{\rm min} \approx \sqrt[3]{\frac{p_\phi^2}{2\rho_c}} \quad \text{and} \quad p_{\rm max} \approx \sqrt{\frac{\kappa}{3}\frac{2}{l_0^2}p_\phi^2} \tag{3.6}$$

JCAP11(2009)014

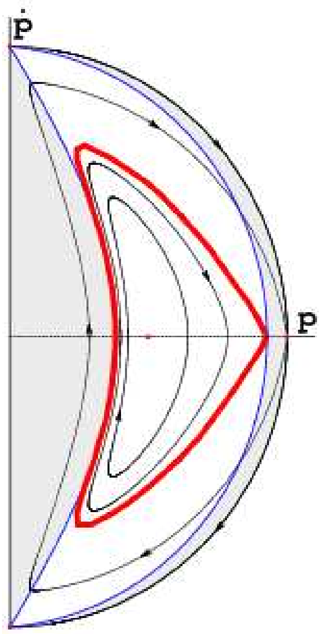


Figure 5. Phase space diagram for the model with free scalar field. The shaded regions are unphysical because $H^2 < 0$ there. Physical solution, on the surface of Hamiltonian constraint $\mathcal{H}_{\rm eff}(p, \dot{p}) \approx 0$, is distinguished by thick (red) line.

based on approximations performed in (2.15) and (2.16). These approximations are valid for $p \gg \frac{\Delta l_0^2}{4} \simeq 18.9\ l_{\rm Pl}^2$.

4 Qualitative analysis of the model with quadratic potential function

Self-interacting scalar fields are commonly used in the modern cosmology. It comes from the fact that violation of the strong energy condition $\rho + 3p \geq 0$ can be obtained with them. Namely $\rho_\phi + 3p_\phi = 2(\dot{\phi}^2 - V(\phi))$ and it is possible to have $V(\phi) > \dot{\phi}^2$ what leads to $\rho + 3p < 0$. In consequence accelerating stage of the evolution of the universe occurs. This mechanism can be therefore applied to modelling dynamics of the inflation or present era of dark energy. Here we concentrate on this former. It is due to the fact that stage of inflation is proceeded by the Planck era and both phases should be dynamically related. Namely initial conditions for the inflation should be given in the Planck epoch. In our approach the Planck epoch is described by the closed loop cosmology. It is interesting to see how phase of inflation emerge in this scenario. In fact this was already studied in case of the flat model [10]. It was shown that phase of bounce can be naturally connected with the inflation. Moreover inflationary attractor is generic and is realised for the broad range of the initial conditions. Here we are going to check whether curvature can affect this scenario.

In our considerations we assume that considered scalar field is massive. Then energy density is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2. \tag{4.1}$$

Dynamical system, in case of self-interacting scalar fields, can be written in the form

$$\dot{\phi} = y, \tag{4.2}$$

$$\dot{y} = -\frac{3}{2}\frac{z}{p}y - \frac{dV(\phi)}{d\phi}, \tag{4.3}$$

$$\dot{p} = z, \tag{4.4}$$

$$\dot{z} = 2\frac{\kappa}{3}\rho_c p \Bigg\{ 2\left(\sin^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) - (1+\gamma^2)\frac{\Delta l_0^2}{4p} + \frac{1}{\rho_c}\left(\frac{1}{2}y^2 + V(\phi)\right)\right) \times \tag{4.5}$$

$$\times \left(\cos^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) + (1+\gamma^2)\frac{\Delta l_0^2}{4p} - \frac{1}{\rho_c}\left(\frac{1}{2}y^2 + V(\phi)\right)\right) \tag{4.6}$$

$$-\frac{1}{2}\left(\frac{1}{2}\sqrt{\frac{\Delta}{p}}l_0 \sin\left(\sqrt{\frac{\Delta}{p}}l_0\right) - (1+\gamma^2)\frac{\Delta l_0^2}{2p} + \frac{3}{\rho_c}y^2\right) \times \tag{4.7}$$

$$\times \left(\cos\left(\sqrt{\frac{\Delta}{p}}l_0\right) + (1+\gamma^2)\frac{\Delta l_0^2}{2p} - \frac{2}{\rho_c}\left(\frac{1}{2}y^2 + V(\phi)\right)\right) \Bigg\} \tag{4.8}$$

together with the constrain

$$\frac{1}{4}z^2 = \frac{\kappa}{3}\rho_c p^2 \left(\sin^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) - (1+\gamma^2)\frac{\Delta l_0^2}{4p} + \frac{\rho_\phi}{\rho_c}\right)\left(\cos^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right) + (1+\gamma^2)\frac{\Delta l_0^2}{4p} - \frac{\rho_\phi}{\rho_c}\right). \tag{4.9}$$

The condition of the positivity of this expression implies that for a constant value of p the motion of the dynamical system in the phase plane $(\phi, \dot{\phi})$ is restricted to

$$\rho_c\left((1+\gamma^2)\frac{\Delta l_0^2}{4p} + \cos^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right)\right) \geq \frac{1}{2}y^2 + \frac{1}{2}m^2\phi^2 \geq \rho_c\left((1+\gamma^2)\frac{\Delta l_0^2}{4p} - \sin^2\left(\sqrt{\frac{\Delta}{p}}\frac{l_0}{2}\right)\right) \tag{4.10}$$

In figure 6 this region is represented by the shaded field (in this example $p = 10$). For a given time region allowed for the motion is in the form of a ring. External edge is due to quantum gravity effects while internal one coming from curvature. Radius of internal edge decreases for increasing p. In figure 6 we show exemplary trajectories on the $(\phi, \dot{\phi})$ phase portrait. Two general types of trajectories are shown there. First type of trajectories is initialised in the point of maximal displacement of the scalar field in the potential, then $\dot{\phi} = 0$. Then filed falls down through the intermediate slow-roll regime, leading to the phase of inflation. This phase of inflation is seen as nearly horizontal attractor line in figure 6. It can be seen also trajectories starting from lower displacements are directed towards this inflationary attractor. These trajectories are shown also in figure 7 where corresponding evolution on the (ϕ, p) plane is presented.

Trajectories of the second kind are initialised in the bottom of the potential, where $\phi = 0$. These trajectories are also directed to the same inflationary attractor. It is worth to analyse this behaviour in more details. Considered evolution is represented by the curves starting at $\phi = 0$ in figure 7. Here initial state is given in the contracting pre-bounce universe. Field is initially in the bottom of the potential well. In this stage, value of the Hubble factor is neglected with respect to m^2. Therefore field undergoes certain oscillations. It can be seen that is this stage field behaves like a dust matter. Universe behaves therefore

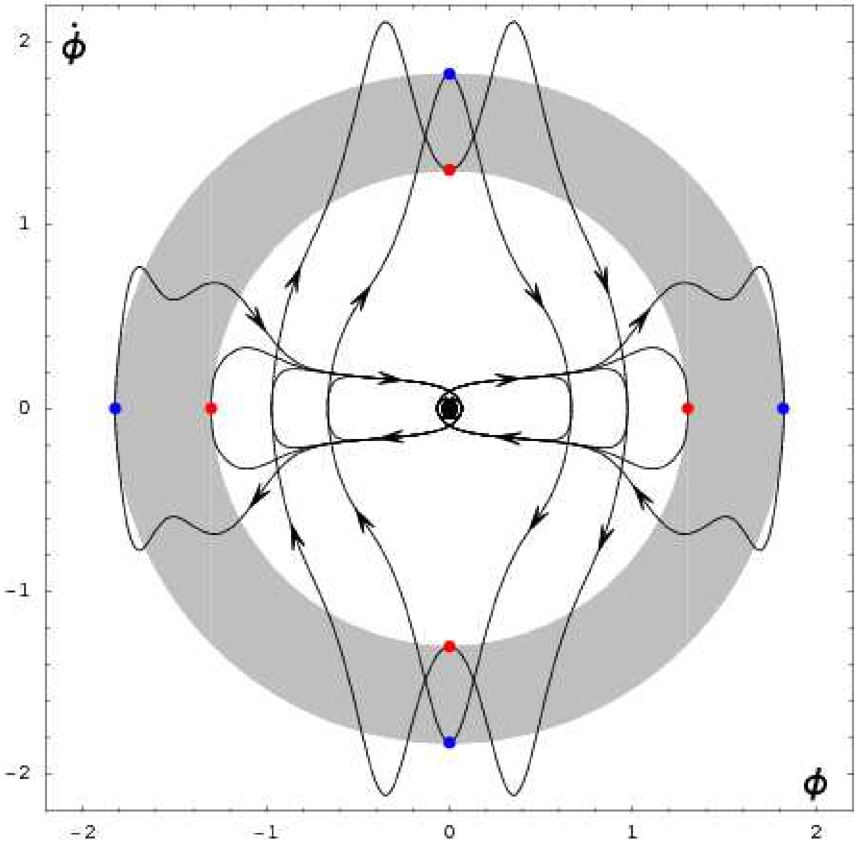


Figure 6. Projection of the $3D$ phase space diagram for the model with the quadratic potential function and $m^2 = 1$ on the $(\phi, \dot{\phi})$ plane. The shaded field denotes region admissible for motion for constant value of $p = 10$, size of this region depends on the value of p. The red and blue dots represent location of the initial conditions taken for numerical simulations.

like contracting, matter dominated one. However when bounce is approached then H grows and becomes significant for the dynamics of ϕ. Since H is negative act as anti-friction term in equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{4.11}$$

driving ϕ up the potential well. For the considered trajectory, field is driven up to $\phi \simeq \pm 1$ in Planck units. Then typical slow-roll inflation takes place leading to exponential growth of p. Field, damped by the friction term, falls down the potential well and starts to oscillate finally. Then again matter-like stage occur. However in this point phase of reheating should stars. This phase however cannot be studied without extension of this model. Therefore physical analysis of dynamics should be stopped at this point.

The important property of the obtained dynamic is that evolution is asymmetric with respect to the bouncing point. Namely the inflation can be realised without any symmetrically situated deflation phase. Moreover phase of the bounce is a dynamical mechanism which can set proper initial conditions for the inflation. Studying the phase trajectories for the different initial conditions one can find how generic is inflationary phase. This issue was recently studied in ref. [21] for the flat LQC with massive scalar field. It was shown that parameters of inflation are only logarithmically sensitive on the initial conditions established in the

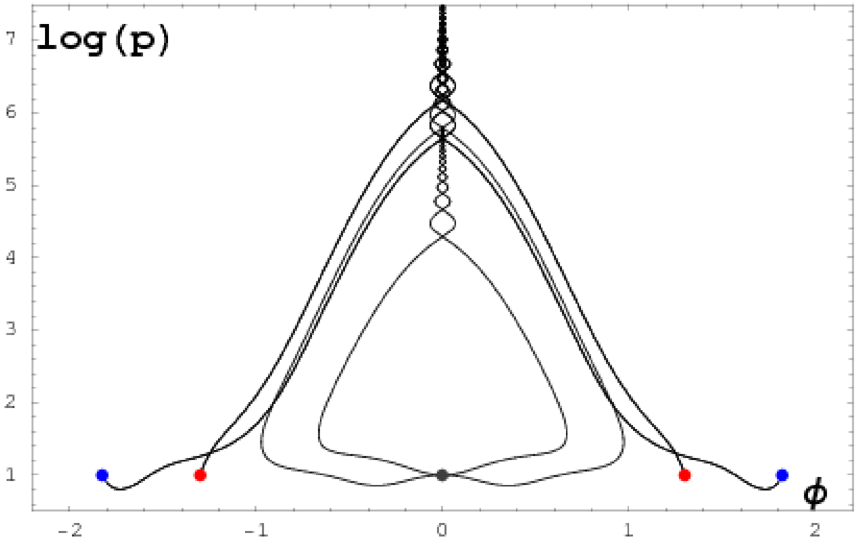


Figure 7. Projection of the $3D$ phase space diagram for the model with the quadratic potential function and $m^2 = 1$ on the $(\phi, \log(p))$ plane. The arrows on the trajectories are omitted due to symmetries (every line can represent both expanding or contracting phase of evolution).

contracting phase. Therefore for the broad rage of initial conditions the similar inflations are obtained. This is very attractive feature of the bouncing cosmologies.

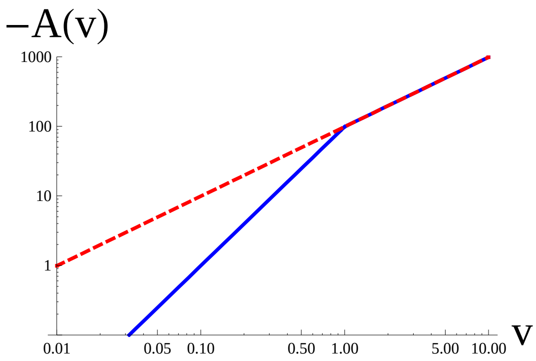
The results from the flat case can be extended to closed model studied here. Namely, we have found that in the neighbourhood of the bouncing point, the influence of curvature term on the dynamics is negligible. Therefore dynamical trajectories are very similar to these obtained in the flat case. Therefore we can conclude that also in the curved model, the proper inflation can be easily obtained. Moreover the fine-tuning of the initial parameters is not required in order to obtain proper amplitude of the primordial perturbations from inflation. Therefore holonomy corrected closed FRW model with a massive scalar field give us good extension of the standard inflationary model where the initial conditions are artificially selected. The advantage of the closed model is also finiteness of the space part. In the flat models the space part is infinite what is rather unphysical and therefore this model can be seen as a sort of approximation. The closed model is in this sense more realistic and therefore worth of exploring also the in context of LQC.

5 Remarks on inverse volume corrections

In the previous sections we have considered models with quantum holonomy corrections. Here we are making some remarks on the possible modifications due to the so called inverse volume corrections. In the flat models definition of these corrections is ambiguous since it deepens on the fiducial cell volume. However in the considered closed model, fiducial cell volume is fixed and inverse volume corrections are well defined. Since effects of inverse volume corrections were extensively studied in literature we are not going to perform full analysis of them. Interested reader can find more information abut them in ref. [6, 22]. In particular interesting feature of the inverse volume corrections is the phase of super-inflation [23, 24].

Here we are going to show where effects of inverse volume corrections start to be im-

JCAP11(2009)014



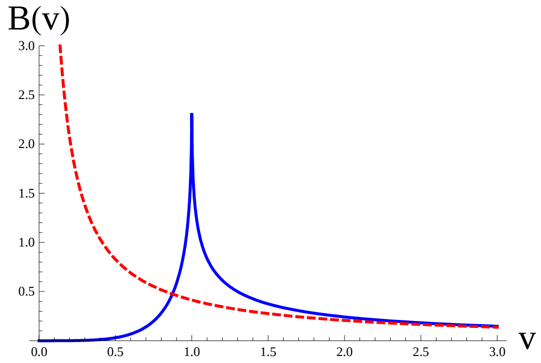


Figure 8. **Left** : Inverse volume correction $A(v)$ (straight line) together with approximation used in this paper (dashed line). **Right** : Inverse volume correction $B(v)$ (straight line) together with approximation used in this paper (dashed line).

portant. We begin this analysis with general effective Hamiltonian derived in ref. [6]

$$\mathcal{H}_{\text{eff}} = \frac{A(v)}{16\pi G}\left[\sin^2\left(\bar{\mu}c - \bar{\mu}\frac{l_0}{2}\right) - \sin^2\left(\bar{\mu}\frac{l_0}{2}\right) + (1+\gamma^2)\frac{\bar{\mu}^2 l_0^2}{4}\right] + \left(\frac{8\pi G\gamma}{6}\right)^{-3/2} B(v)\frac{p_\phi^2}{2}. \tag{5.1}$$

Here functions $A(v)$ and $B(v)$ contains inverse volume corrections and are given by

$$A(v) = -\frac{27K}{4\gamma^{3/2}}\sqrt{\frac{8\pi}{6}}|v|\,||v-1|-|v+1||\,, \tag{5.2}$$

$$B(v) = \left(\frac{3}{2}\right)^3 K|v|\left||v+1|^{1/3} - |v-1|^{1/3}\right|^3 \tag{5.3}$$

where

$$K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}. \tag{5.4}$$

For $v \gg 1$ functions can be expanded as follows

$$A(v) \simeq -\frac{2\sqrt{48\pi v}}{\gamma^{3/2}3\sqrt{3}K}, \tag{5.5}$$

$$B(v) \simeq \frac{K}{v}. \tag{5.6}$$

We show both functions (5.2) and (5.3) with corresponding approximations (5.5) and (5.6) in figure 8. Approximations (5.5) and (5.6) are precisely those applied in this paper, leading to initial Hamiltonian (2.1). Form figure 8 we see that corrections become important below $v \approx 2$. Since relation between p and v variables is given by

$$p(v) = \frac{\kappa\gamma}{6}\left(\frac{v}{K}\right)^{2/3} \tag{5.7}$$

we find $p(2) \simeq 2.8$. Therefore for $p \lesssim 3$ inverse volume corrections becomes significant. So we have to be aware that results obtained in this paper are valid for $p \gtrsim 3$ and below this value

JCAP11(2009)014

other quantum effects should take place. However only small fraction of trajectories probe this region. In fact we can always choose sufficient initial conditions to avoid this region. For instance for the model with a free scalar field, sufficiently high value of p_ϕ has to be chosen. Namely condition for $p > 3$ gives $p_\phi > 18\ l_{\rm Pl}$.

6 Summary

In the classical cosmology positive curvature introduces $-1/a^2$ term in the Friedmann equation. In the early universe other terms like radiation $1/a^4$ dominate and dynamical effect of curvature is negligible. However also quantum gravitational effects start to be important then and dynamics should be modified. Loop quantum cosmology gives us opportunity to study these effects. In particular in the effective formulation of LQC dynamics is governed by the classical equations with proper quantum corrections. These corrections modify also the curvature term $-1/a^2$ and therefore dynamical significance of this term can be changed in the quantum regime. In this paper we have investigated whether curvature term could play significant role in early universe due to the quantum effects. We have found that curvature does not affect evolution in the quantum regime. In particular, maximal value of the Hubble factor is the same like in the flat case, namely $H_{\rm max} = \sqrt{\frac{\kappa}{12}\rho_c}$.

In our considerations we concentrated on the models with cosmological constant, free scalar field and massive scalar field. In the models with cosmological constant and free scalar field, dynamics reduce to 2D system. For the free field case, universe undergoes non-singular oscillations. This behaviour does not depend on the value of parameter p_ϕ. However for the model with cosmological constant only, type of solution depends on the value of Λ. For $\Lambda < \Lambda_{\rm c}$ bouncing solutions with asymptotic de Sitter stages are obtained. While $\Lambda > \Lambda_{\rm c}$ we find non-singular oscillations. Case $\Lambda = \Lambda_{\rm c}$ is intermediate state when upper boundary reaches infinity. Here Λ_c is critical value of Λ and is given by $\Lambda_{\rm c} = \frac{\sqrt{3}m_{\rm Pl}^2}{2\pi\gamma^3} \simeq 21 m_{\rm Pl}^2$. Emergence of this value is due to quantum modifications of the dynamics. Therefore observed change of the type of solution is of the quantum origin and does not appear in the classical case.

Physically interesting consequences occur in the models with self-interacting scalar field. Namely, the point is that, bounce gives mechanism to set good initial conditions for inflation. During the bounce field is driven up the potential well what is necessary to start phase of inflation. In the classical theory of inflation initial conditions have to be fixed suitably. Here they can be obtained from the different pre-bounce evolutions without need of any precise adjustment. This is nice feature of the bounce stage and is now confirmed for the flat and closed models in LQC.

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JCAP11(2009)014