Niegeometryczny opis grawitacji

Równania Einsteina bez geometrii różniczkowej

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Literatura

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Elektrodynamika

$$S[A^\mu] = \int d^4x \mathcal{L}_{EM}$$

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Czyli

$$S[A^\mu] = \int d^4x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_\mu A^\mu\right]$$

Równania pola

$$\Box A^\mu - \partial^\mu(\partial_\alpha A^\alpha) = -j^\mu$$

Możliwe człony kinematyczne

1. $h_{\mu\nu,\sigma}h^{\mu\nu,\sigma}$
2. $h_{\mu\nu,\sigma}h^{\mu\sigma,\nu}$
3. $h^{\mu\nu}{}_{,\nu}h^{\sigma}{}_{\mu,\sigma}$
4. $h^{\mu\nu}{}_{,\nu}h^{\sigma}{}_{\sigma\mu}$
5. $h^{\mu}{}_{\nu,\mu}h_{\sigma}{}^{\sigma,\mu}$

Wyrazy 2 i 3 są równoważne

$$2.\ h^{\mu\nu}{}_{,\sigma}h^{\sigma}{}_{\mu,\nu} = (h^{\mu\nu}h^{\sigma}{}_{\mu,\nu})_{,\sigma} - h^{\mu\nu}h^{\sigma}{}_{\mu,\nu\sigma}$$
$$3.\ h^{\mu\nu}{}_{,\nu}h^{\sigma}{}_{\mu,\sigma} = (h^{\mu\nu}h^{\sigma}{}_{\mu,\sigma})_{,\nu} - h^{\mu\nu}h^{\sigma}{}_{\mu,\sigma\nu}$$

$$S[h_{\mu\nu}] = \int d^4x \left(a h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + b h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma} + \right.$$
$$\left. c h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu} + d h^{\mu}{}_{\nu,\mu} h_{\sigma}{}^{\sigma,\mu} - \lambda T^{\mu\nu} h_{\mu\nu} \right)$$

Zasada stacjonarności działania $\delta S[h_{\mu\nu}] = 0$
Pole $h_{\mu\nu}$ jest symetryczne, musimy odpowiednio zdefiniować pochodną wariacyjną

$$\frac{\delta h_{\alpha\beta}}{\delta h_{\mu\nu}} = \frac{1}{2}\left[\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\right] \equiv \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} = \delta^{(\alpha}_{\mu}\delta^{\beta)}_{\nu}$$
$$\frac{\delta h_{\alpha\beta,\gamma}}{\delta h_{\mu\nu,\sigma}} = \frac{1}{2}\left[\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\sigma} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\delta^{\gamma}_{\sigma}\right] \equiv \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\delta^{\gamma}_{\sigma} = \delta^{(\alpha}_{\mu}\delta^{\beta)}_{\nu}\delta^{\gamma}_{\sigma}$$

Wariacja członu a

$$\delta(h_{\alpha\beta,\gamma}h^{\alpha\beta,\gamma}) = \eta^{\alpha\delta}\eta^{\beta\kappa}\eta^{\gamma\xi}\delta(h_{\alpha\beta,\gamma}h_{\delta\kappa,\xi}) =$$
$$\eta^{\alpha\delta}\eta^{\beta\kappa}\eta^{\gamma\xi}\left[\frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\gamma\sigma}+\delta_{\alpha\nu}\delta_{\beta\mu}\delta_{\gamma\sigma})h_{\delta\kappa,\xi}+\right.$$
$$\left.\frac{1}{2}(\delta_{\delta\mu}\delta_{\kappa\nu}\delta_{\xi\sigma}+\delta_{\delta\nu}\delta_{\kappa\mu}\delta_{\xi\sigma})h_{\alpha\beta,\gamma}\right]\delta h_{\mu\nu,\sigma} =$$
$$= \frac{1}{2}\left[\eta^{\mu\delta}\eta^{\nu\kappa}\eta^{\sigma\xi}h_{\delta\kappa,\xi}+\eta^{\nu\delta}\eta^{\mu\kappa}\eta^{\sigma\xi}h_{\delta\kappa,\xi}+\right.$$
$$\left.+\eta^{\mu\delta}\eta^{\beta\nu}\eta^{\gamma\sigma}h_{\alpha\beta,\gamma}+\eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\gamma\sigma}h_{\alpha\beta,\gamma}\right]\delta h_{\mu\nu,\sigma} =$$
$$= 2h_{\mu\nu,\sigma}\delta h_{\mu\nu,\sigma}$$

Wariacja członu d

$$\delta(h^{\alpha}_{\alpha,\beta}h^{\gamma,\beta}_{\gamma}) = \eta^{\alpha\delta}\eta^{\gamma\kappa}\eta^{\beta\xi}\delta(h_{\delta\alpha,\beta}h_{\gamma\kappa,\xi}) =$$

$$\eta^{\alpha\delta}\eta^{\gamma\kappa}\eta^{\beta\xi}\left[\frac{1}{2}(\delta_{\delta\mu}\delta_{\alpha\nu}\delta_{\beta\sigma}+\delta_{\delta\nu}\delta_{\alpha\mu}\delta_{\beta\sigma})h_{\gamma\kappa,\xi}+\right.$$

$$\left.\frac{1}{2}(\delta_{\gamma\mu}\delta_{\kappa\nu}\delta_{\xi\sigma}+\delta_{\gamma\nu}\delta_{\kappa\mu}\delta_{\xi\sigma})h_{\delta\alpha,\beta}\right]\delta h_{\mu\nu,\sigma} =$$

$$= \frac{1}{2}\left[\eta^{\mu\nu}\eta^{\gamma\kappa}\eta^{\sigma\xi}h_{\gamma\kappa,\xi}+\eta^{\nu\mu}\eta^{\gamma\kappa}\eta^{\sigma\xi}h_{\gamma\kappa,\xi}+\right.$$

$$\left.+\eta^{\alpha\delta}\eta^{\mu\nu}\eta^{\xi\sigma}h_{\delta\alpha,\xi}+\eta^{\alpha\delta}\eta^{\nu\mu}\eta^{\beta\sigma}h_{\delta\alpha,\beta}\right]\delta h_{\mu\nu,\sigma} =$$

$$= 2\eta^{\mu\nu}h^{\gamma,\sigma}_{\gamma}\delta h_{\mu\nu,\sigma}$$

Dostajemy rówanie

$$2ah^{\mu\nu,\sigma}_{,\sigma}+b(h^{\mu,\nu\sigma}_{\sigma}+h^{\nu,\mu\sigma}_{\sigma})+c(h^{\sigma,\mu\nu}_{\sigma}+\eta^{\mu\nu}h^{\gamma\lambda}_{,\lambda\gamma})+2d\eta^{\mu\nu}h^{\gamma,\lambda}_{\gamma,\lambda}=-\lambda T^{\mu\nu}$$

Zasada zachowania ładunku w elektrodynamice

$$j^{\mu}{}_{,\mu}=0$$

jest konsekwencją równań pola.
Chcemy aby energia była zachowana

$$T^{\mu\nu}{}_{,\nu}=0$$

$$
\begin{aligned}
2ah^{\mu\nu,\sigma}_{,\sigma\nu} + bh^{\mu\sigma,\nu}_{,\sigma\nu} + bh^{\nu\sigma,\mu}_{,\sigma\nu} &\quad + \\
ch^{\sigma,\mu\nu}_{\sigma,\nu} + ch^{\gamma\alpha,\mu}_{,\gamma\alpha} + 2dh^{\sigma,\gamma\mu}_{\sigma,\nu} &= \quad 0
\end{aligned}
$$

Biorąc $a = \frac{1}{2}$ mamy

$$
\begin{aligned}
h^{\mu\nu,\sigma}{}_{,\sigma\nu}(2a+b) &= 0 \\
h^{\nu\sigma,\mu}{}_{,\nu\sigma}(b+c) &= 0 \\
h^{\sigma}{}_{\sigma,\nu}{}^{,\mu\nu}(c+2d) &= 0
\end{aligned}
\qquad
\begin{aligned}
a &= \frac{1}{2} \\
b &= -1 \\
c &= 1 \\
d &= -\frac{1}{2}
\end{aligned}
$$

Równanie ruchu:

$$h^{\mu\nu,\sigma}_{,\sigma} - h^{\mu,\nu\sigma}_{\sigma} - h^{\nu,\mu\sigma}_{\sigma} + h^{\sigma,\mu\nu}_{\sigma} + \eta^{\mu\nu} h^{\gamma\lambda}_{,\lambda\gamma} - \eta^{\mu\nu} h^{\gamma,\lambda}_{\gamma,\lambda} = -\lambda T^{\mu\nu}$$

Oznaczając

$$\bar{X}_{\mu\nu} = \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu}) - \frac{1}{2}\eta_{\mu\nu} X^{\sigma}_{\sigma}$$

mamy

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\sigma}_{\sigma}$$

$$\bar{\bar{h}}_{\mu\nu} = h$$

W tej notacji równania pola maja postać

$$h_{\mu\nu,\sigma}{}^{,\sigma} - 2\bar{h}_{\mu\sigma,\nu}{}^{,\sigma} = -\lambda \bar{T}_{\mu\nu}$$

Elektrodynamika

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\tilde{A}^\mu = A^\mu + \partial^\mu f$$

$$\Box A^\mu - \partial^\mu(\partial_\alpha A^\alpha) = -j^\mu$$

Cechowanie Lorentza $\partial_\alpha A^\alpha = 0$

$$\begin{aligned} 0 = \partial_\mu \tilde{A}^\mu &= \partial_\mu A^\mu + \Box f \\ \Box f &= -\partial_\mu A^\mu \\ \Box A^\mu &= -j^\mu \end{aligned}$$

Grawitacja

$$h_{\mu\nu,\sigma}{}^{,\sigma} - 2\bar{h}_{\mu\sigma,\nu}{}^{,\sigma} = -\lambda \bar{T}_{\mu\nu}$$

jest niezmiennicze wzgledem transformacji

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + X_{\mu,\nu} + X_{\nu,\mu}$$

Cechowanie Lorentza $\bar{h}^{\mu\nu}{}_{,\nu} = 0$

$$\begin{aligned} 0 = \partial_\mu \tilde{\bar{h}}^{\mu\nu} &= \partial_\mu \bar{h}^{\mu\nu} + \partial_\mu \bar{X}^{\mu,\nu} + \Box \bar{X}^\nu \\ \Box X^\mu &= -\partial_\nu \bar{h}^{\nu\mu} \\ \Box \bar{h}_{\mu\nu} &= -\lambda T_{\mu\nu} \end{aligned}$$

Rozwiązania próżniowe

Elektrodynamika

$$\begin{aligned} \Box A_\mu &= 0 \\ A_\mu &= C_\mu e^{ikx} \\ k_\nu k^\nu C_\mu e^{ikx} &= 0 \end{aligned}$$

Warunek $\partial_\alpha A^\alpha = 0$ daje

$$k^\mu C_\mu = 0$$

Resztkowa swoboda cechowania

$$\begin{aligned} f \to f &+ a \\ \Box a &= 0 \end{aligned}$$

Cechowanie czasowe $A_0 = 0$
2 stopnie swobody pola

Grawitacja

$$\begin{aligned} \Box \bar{h}_{\mu\nu} &= 0 \\ \bar{h}_{\mu\nu} &= C_{\mu\nu} e^{ikx} \\ k_\nu k^\nu &= 0 \\ k^\mu C_{\mu\nu} &= 0 \end{aligned}$$

10-4 stopni swobody
Resztkowa swoboda cechowania

$$\begin{aligned} C_{0\mu} &= 0 \\ C^\mu{}_{,\mu} &= 0 \end{aligned}$$

2 stopnie swobody

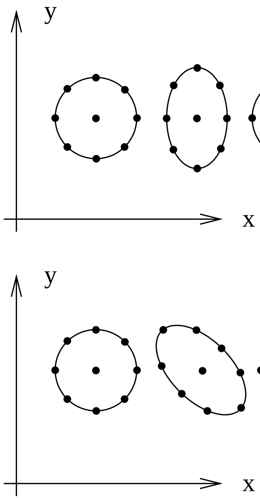
Fale grawitacyjne

$$C_{\mu\nu} = C_+ e^1_{\mu\nu} + C_\times e^2_{\mu\nu}$$

$$e^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} C_R &= \frac{1}{\sqrt{2}}(C_+ + iC_\times) \\ C_L &= \frac{1}{\sqrt{2}}(C_+ - iC_\times) \end{aligned}$$


y
x
y
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$$S = \int d\alpha \left[-\frac{m_0}{2} \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx_\mu}{d\alpha} \right) - eA_\mu \left(\frac{dx^\mu}{d\alpha} \right) \right]$$

$$m_0 \frac{d^2 x_\mu}{d\alpha^2} = eF_{\mu\nu} \left(\frac{dx^\nu}{d\alpha} \right)$$

Grawitacja

$$T^{\mu\nu} = m_0 \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right)$$

$$S = \int d\alpha \left[-\frac{m_0}{2} \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx_\mu}{d\alpha} \right) - \lambda h_{\mu\nu} m_0 \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right) \right]$$

$$S = \int d\alpha \left[-\frac{m_0}{2} \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx_\mu}{d\alpha} \right) - \lambda h_{\mu\nu} m_0 \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right) \right]$$

$$S = \int d\alpha \left[-\frac{m_0}{2} (\eta_{\mu\nu} + 2\lambda h_{\mu\nu}) \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right) \right]$$

Oznaczamy $g_{\mu\nu} = \eta_{\mu\nu} + 2\lambda h_{\mu\nu}$. Czyli

$$S = \int d\alpha \left[-\frac{m_0}{2} g_{\mu\nu}(x) \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right) \right]$$

$$S = \int d\alpha \left[-\frac{m_0}{2} g_{\mu\nu}(x) \left(\frac{dx^\mu}{d\alpha} \right) \left(\frac{dx^\nu}{d\alpha} \right) \right]$$

Wariujemy działanie, $\delta S[x^\mu] = 0$
Wprowadzamy oznaczenie

$$[\mu\nu, \sigma] = \frac{1}{2} \left[g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right]$$

Otrzymujemy równanie ruchu

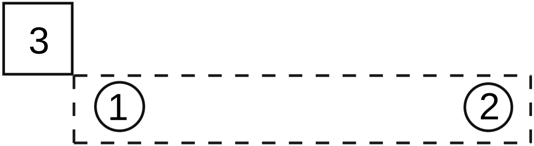
$$g_{\sigma\nu} \ddot{x}^\nu + [\mu\nu, \sigma] \dot{x}^\mu \dot{x}^\nu = 0$$

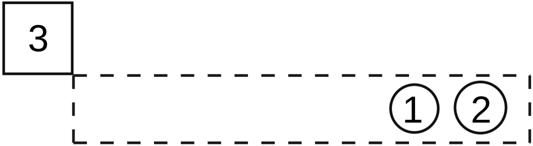
Energia własna pola grawitacyjnego

Uwzględniamy nieliniowość teorii
Nasza poprzednia definicja tensora energii-pędu była niepoprawna

$$({}^{\circ}T^{\mu\nu} + \chi^{\mu\nu})_{,\nu}$$

$$T^{\mu\nu} =^{\circ} T^{\mu\nu} + \chi^{\mu\nu}$$





$$^{\circ}T^{\mu\nu}(x) = m_0 \int ds \delta^4(x - z(s))\dot{z}^{\mu}\dot{z}^{\nu}$$

$$\begin{aligned} ^{\circ}T^{\mu\nu}(x)_{,\nu} &= m_0 \int ds \frac{\partial \delta^4(x-z(s))}{\partial x^{\nu}} \dot{z}^{\mu}\dot{z}^{\nu} = \\ -m_0 \int ds \frac{\partial \delta^4(x-z(s))}{\partial z^{\nu}} \dot{z}^{\mu}\dot{z}^{\nu} &= -m_0 \int ds \frac{d\delta^4(x-z(s))}{ds} \dot{z}^{\mu} = \\ \underbrace{-m_0 \delta^4(x-z(s))\dot{z}^{\mu}|_{-\infty}^{+\infty}}_{=0} &+ m_0 \int ds \delta^4(x-z(s))\ddot{z}^{\mu} \end{aligned}$$

$$g_{\mu\nu}\ddot{z}^{\mu} = -[\mu\nu, \lambda]\dot{z}^{\mu}\dot{z}^{\nu}$$

$$
\begin{aligned}
{}^\circ T^{\mu\nu}(x)_{,\nu} &= m_0 \int ds \delta^4(x - z(s)) \ddot{z}^\mu \\
{}^\circ T^{\mu\nu}(x)_{,\nu} g_{\mu\lambda}(x) &= m_0 \int ds \delta^4(x - z(s)) \underbrace{g_{\mu\lambda}(z)\ddot{z}^\mu}_{-[\mu\nu,\lambda]_z \dot{z}^\mu \dot{z}^\nu} \\
{}^\circ T^{\mu\nu}(x)_{,\nu} g_{\mu\lambda}(x) &= -[\mu\nu,\lambda]_x m_0 \int ds \delta^4(x - z(s)) \dot{z}^\mu \dot{z}^\nu \\
{}^\circ T^{\mu\nu}(x)_{,\nu} g_{\mu\lambda}(x) &= -[\mu\nu,\lambda] {}^\circ T^{\mu\nu}(x)
\end{aligned}
$$

Teoria poprawna we wszystkich rzędach

Chcemy aby

$$\frac{\delta F}{\delta h_{\mu\nu}} = \lambda T^{\mu\nu}$$

oraz

$$T^{\mu\nu}_{,\nu} g_{\mu\lambda} = -[\mu\nu, \lambda] T^{\mu\nu}$$

Czyli musimy rozwiazać równanie funkcjonalne

$$\left(\frac{\delta F}{\delta h_{\mu\nu}}\right)_{,\nu} g_{\mu\lambda} + [\mu\nu, \lambda] \left(\frac{\delta F}{\delta h_{\mu\nu}}\right) = 0$$

$$\int d^4x \left[-A^\lambda \left(\frac{\delta F}{\delta h_{\mu\nu}} \right)_{,\nu} g_{\mu\lambda} + A^\lambda[\mu\nu,\lambda] \left(\frac{\delta F}{\delta h_{\mu\nu}} \right) \right] = 0$$

$$\int d^4x \left(\frac{\delta F}{\delta h_{\mu\nu}} \right) \mathrm{Sym} \left[\left(A^\lambda g_{\mu\lambda} \right)_{,\nu} + A^\lambda[\mu\nu,\lambda] \right] = 0$$

Rozwińmy funkcjonał

$$F[h_{\mu\nu} + \xi_{\mu\nu}] = F[h_{\mu\nu}] + \xi_{\mu\nu} \left(\frac{\delta F}{\delta h_{\mu\nu}} \right) + \mathcal{O}(\xi^2)$$

czyli

$$\xi_{\mu\nu} = \mathrm{Sym} \left[- \left(A^\lambda g_{\mu\lambda} \right)_{,\nu} + A^\lambda[\mu\nu,\lambda] \right]$$

$$\xi_{\mu\nu} = \text{Sym}\left[-A^{\lambda}_{\nu} g_{\mu\lambda} - A^{\lambda}_{\nu} g_{\mu\lambda,\nu} + \frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right) A^{\lambda}\right]$$

$$\xi_{\mu\nu} = \frac{1}{2}\left(-g_{\lambda\mu} A^{\lambda}_{,\nu} - A^{\lambda}_{,\mu} g_{\lambda\nu} - g_{\mu\nu,\lambda} A^{\lambda}\right)$$

Pamiętamy że $g_{\mu\nu} = \eta_{\mu\nu} + 2\lambda h_{\mu\nu}$. Czyli

$$g'_{\mu\nu} = g_{\mu\nu} + 2\lambda\xi_{\mu\nu} = g_{\mu\nu} + 2\lambda\frac{1}{2}\left(-g_{\lambda\mu} A^{\lambda}_{,\nu} - A^{\lambda}_{,\mu} g_{\lambda\nu} - g_{\mu\nu,\lambda} A^{\lambda}\right)$$

Przyjmując $\zeta^{\mu} = -\lambda A^{\mu}$ dostajemy

$$g'_{\mu\nu} = g_{\mu\nu} + g_{\mu\lambda}\zeta^{\lambda}_{\ ,\nu} + g_{\nu\lambda}\zeta^{\lambda}_{\ ,\mu} + g_{\mu\nu,\lambda}\zeta^{\lambda}$$

Badamy niezmienniczość wyznacznika

$$
\begin{aligned}
g'_{\mu\nu} &= g_{\mu\nu} + g_{\mu\lambda}\zeta^{\lambda}{}_{,\nu} + g_{\nu\lambda}\zeta^{\lambda}{}_{,\mu} + g_{\mu\nu,\lambda}\zeta^{\lambda} \\
g'_{\mu\nu} &= g_{\mu\beta}\left[\delta^{\beta}_{\nu} + \left(g^{\beta\gamma}g_{\gamma\lambda}\zeta^{\lambda}{}_{,\nu} + g^{\beta\gamma}g_{\nu\lambda}\zeta^{\lambda}{}_{,\gamma} + g^{\beta\gamma}g_{\gamma\nu,\lambda}\zeta^{\lambda}\right)\right]
\end{aligned}
$$

Korzystamy z własności $\det(1+A) = 1 + \mathrm{tr}(A) + \mathcal{O}(A^2)$.

$$
\det g' = \det g[1 + \mathrm{tr}(\dots)]
$$

gdzie

$$
\begin{aligned}
\mathrm{tr}(\dots) &= g^{\beta\gamma}g_{\gamma\lambda}\zeta^{\lambda}{}_{,\beta} + g^{\beta\gamma}g_{\beta\lambda}\zeta^{\lambda}{}_{,\gamma} + g^{\beta\gamma}g_{\gamma\beta,\lambda}\zeta^{\lambda} = \\
&= 2\zeta^{\alpha}_{\alpha} + g^{\beta\gamma}g_{\gamma\beta,\lambda}\zeta^{\lambda}
\end{aligned}
$$

$$\ln\left(-\det g'\right) = \ln\left(-\det g[1+\mathrm{tr}(\dots)]\right) =$$
$$= \ln\left(-\det g\right) + 2\zeta^{\alpha}_{,\alpha} + g^{\beta\gamma}g_{\gamma\beta,\lambda}\zeta^{\lambda}$$

Pomocnicze wyrażenie

$$g_{\mu\nu}(x+dx) = g_{\mu\nu}(x) + dx^{\alpha}g_{\mu\nu,\alpha} = g_{\mu\beta}(\delta^{\beta}_{\nu} + g^{\beta\gamma}g_{\gamma\nu,\alpha}dx^{\alpha})$$
$$\det g_{\mu\nu}(x+dx) = \det g_{\mu\nu}(x)\left(1 + g^{\beta\gamma}g_{\gamma\beta,\alpha}dx^{\alpha}\right)$$
$$(\det g)_{,\alpha} = \det g\; g^{\beta\gamma}g_{\gamma\beta,\alpha}$$
$$(ln(-\det g))_{,\alpha} = g^{\beta\gamma}g_{\gamma\beta,\alpha}$$

$$\ln\left(-\det g'\right) = \ln\left(-\det g\right) + 2\zeta^{\alpha}_{,\alpha} + (ln(-\det g))_{,\lambda}\zeta^{\lambda}$$

Wprowadzając oznaczenie $C = \ln\left(-\det g\right)$ dostajemy

$$C' = C + 2\zeta^{\alpha}_{,\alpha} + C_{,\lambda}\zeta^{\lambda}$$

$$\begin{aligned} e^{aC'} &= e^{a(2\zeta^{\alpha}_{,\alpha} + C_{,\lambda}\zeta^{\lambda})} = e^{aC}(1 + 2a\zeta^{\alpha}_{,\alpha} + aC_{,\lambda}\zeta^{\lambda}) + \mathcal{O}(\zeta^2) \\ &= e^{aC} + 2a \underbrace{e^{aC}\zeta^{\alpha}_{,\alpha}}_{\left(e^{aC}\zeta^{\alpha}\right)_{,\alpha} - aC_{\lambda}\zeta^{\lambda}e^{aC}} + ae^{aC}C_{,\lambda}\zeta^{\lambda} + \mathcal{O}(\zeta^2) \end{aligned}$$

Czyli biorąc $a = \frac{1}{2}$ mamy

$$e^{C'/2} = e^{C/2} + \left(e^{aC}\zeta^{\alpha}\right)_{,\alpha}$$

$$e^{C'/2} = e^{C/2} + \left(e^{aC}\zeta^{\alpha}\right)_{,\alpha}$$

$$\int d^4x e^{C'/2} = \int d^4x e^{C/2}$$

Czyli pamiętając że $C = \ln\left(-\det g\right)$ mamy

$$^{\circ}F = \int d^4x \sqrt{-\det g}$$

Lub

$$F_{\Lambda} = \frac{1}{2\lambda^2} \int d^4x \sqrt{-\det g}\ \Lambda$$

Lagranżjan pełnej teorii

$$F = -\frac{1}{2\lambda^2}\int d^4x g^{\mu\nu} R^{\tau}{}_{\mu\nu\tau}\sqrt{-\det g}$$

Wariując

$$2\frac{\delta F}{\delta g_{\mu\nu}} = -\frac{1}{\lambda^2}\frac{\delta\left(\sqrt{-\det g}R\right)}{\delta g_{\mu\nu}} = \frac{1}{\lambda^2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)$$

Definiując $\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\det g}}\frac{\delta F}{\delta g_{\mu\nu}}$, oraz $\lambda = \sqrt{8\pi G}$ dostajemy

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G\tilde{T}^{\mu\nu}$$