Cosmic variance

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Let us consider random variable y with distribution

$$P(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}},$$

where

$$\int_{-\infty}^{+\infty} dy P(y) = 1.$$

The average is defined as follows

$$\langle ... \rangle := \int_{-\infty}^{+\infty} dy ... P(y).$$

The odd moments are:

$$\langle y^{2n+1} \rangle = \int_{-\infty}^{+\infty} dy \; y^{2n+1} P(y) = 0.$$

The even moments are:

$$\langle y^{2n} \rangle = \int_{-\infty}^{+\infty} dy \; y^{2n} P(y) = \sigma^{2n}(2n-1)\cdot(2n-3)\dots 5\cdot 3\cdot 1.$$

We have $2n$ points. One point can be connected with one of the remaining $(2n-1)$. One of the remaining points can be connected with another $(2n-3)$, and so on... Based on this observation one can write

$$\langle yy \rangle = \sigma^2$$

and then

$$\langle yyyy \rangle = \langle yy \rangle\langle yy \rangle + \langle yy \rangle\langle yy \rangle + \langle yy \rangle\langle yy \rangle = 3\sigma^4.$$

This construction is called a Wick's theorem.

Now we have n independent Gaussian random variables $x_1, x_2, ..., x_n$, where

$$P(x_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_i^2}{2\sigma^2}}.$$

The mean value is defined as follows

$$\langle ... \rangle = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} dx_1 ... dx_n ... P(x_1) ... P(x_n)$$

Based on this one can determine

$\langle x_i \rangle = 0$

$\langle x_i^2 \rangle = \sigma^2$

$\langle x_i^3 \rangle = 0$

$\langle x_i^4 \rangle = 3\sigma^4$

$\langle x_i^2 x_j^2 \rangle = \langle x_i x_i \rangle \langle x_j x_j \rangle + \langle x_i x_j \rangle \langle x_i x_j \rangle + \langle x_i x_j \rangle \langle x_i x_j \rangle = \langle x_i^2 \rangle \langle x_j^2 \rangle =$
$\sigma^2 \sigma^2 = \sigma^4$

Now let us consider variable

$$x = \frac{1}{n}\sum_{i=1}^{n} x_i^2.$$

We are going to derive variance of x:

$$\mathrm{var}[x] = \langle x^2 \rangle - \langle x \rangle^2.$$

We have

$$\langle x \rangle = \frac{1}{n}\sum_{i=1}^{n}\langle x_i^2 \rangle = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \sigma^2.$$

Therefore

$$\langle x \rangle^2 = \sigma^4.$$

$$x^2 = \frac{1}{n^2}\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} x_j^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^4 + \sum_{i\neq j} x_i^2 x_j^2\right).$$

$$\langle x^2\rangle = \frac{1}{n^2}\left(\sum_{i=1}^{n}\langle x_i^4\rangle + \sum_{i\neq j}\langle x_i^2 x_j^2\rangle\right) = \frac{n3\sigma^4 + (n^2-n)\sigma^4}{n^2} = \frac{2\sigma^4}{n} + \sigma^4.$$

$$\mathrm{var}[x] = \langle x^2\rangle - \langle x\rangle^2 = \frac{2\sigma^4}{n} + \sigma^4 - \sigma^4 = \frac{2\sigma^4}{n}.$$

$$D[x] = \sqrt{\mathrm{var}[x]} = \sqrt{\frac{2}{n}}\sigma^2.$$

CMB anisotropy

In order to describe fluctuation of temperature we define

$$\Theta(\hat{n}, x, \eta) := \frac{\Delta T}{T}.$$

This is a function of position is space x, conformal time η and direction on sky $\hat{n}$. However, we can measure this quantity only at some x_0 and η_0. This will be the origin of so-called *cosmic variance*.
The $\Theta(\hat{n}, x, \eta)$ can be decomposed in terms of spherical harmonics

$$\Theta(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_l^m(\hat{n}).$$

Because $\Theta(\hat{n})^* = \Theta(\hat{n})$, with use of relation $Y_l^m(\hat{n})^* = Y_l^{-m}(\hat{n})$, we find

$$a_{lm}^* = a_{l-m}.$$

For the Gaussian random field:

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l.$$

The C_l is called a spectrum of anisotropy. However, because the average over different positions cannot be performed, this quantity cannot be measured. What is measured instead is the average:

$$C_l^{\text{obs}} = \frac{1}{2l+1} \sum_{m=-l}^{l} a_{lm} a_{l-m}.$$

Spatial averaging of this quantity gives

$$\langle C_l^{\text{obs}} \rangle = \frac{1}{2l+1} \sum_{m=-l}^{l} \langle a_{lm} a_{l-m} \rangle = \frac{1}{2l+1} \sum_{m=-l}^{l} C_l = C_l.$$

We are interested in computing the quantity:

$$\frac{\Delta C_l}{C_l} := \frac{\sqrt{\langle (C_l - C_l^{\text{obs}})^2 \rangle}}{C_l},$$

which quantify how measurements of C_l^{obs} recover the spectrum C_l. We have

$$\langle \left(C_l - C_l^{\text{obs}} \right)^2 \rangle = C_l^2 - 2C_l \langle C_l^{\text{obs}} \rangle + \langle (C_l^{\text{obs}})^2 \rangle = -C_l^2 + \langle (C_l^{\text{obs}})^2 \rangle,$$

where

$$\langle (C_l^{\text{obs}})^2 \rangle = \frac{1}{(2l+1)^2} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \langle a_{lm} a_{l-m} a_{lm'} a_{l-m'} \rangle.$$

Wick's theorem:

For the Gaussian random fields $\phi_1, \phi_2, ...\phi_n$ we have

$$\langle \phi_1 \phi_2 ... \phi_n \rangle = \sum_{\text{pairing}} \prod_{\text{pairs}} \langle \phi_i \phi_j \rangle$$

Based this we obtain

$$\begin{aligned} \langle a_{lm} a_{l-m} a_{lm'} a_{l-m'} \rangle &= \langle a_{lm} a_{l-m} \rangle \langle a_{lm'} a_{l-m'} \rangle \\ &+ \langle a_{lm} a_{lm'} \rangle \langle a_{l-m} a_{l-m'} \rangle \\ &+ \langle a_{lm} a_{l-m'} \rangle \langle a_{l-m} a_{lm'} \rangle . \end{aligned}$$

$$\sum_{m=-l}^{l} \sum_{m'=-l}^{l} \langle a_{lm} a_{l-m} \rangle \langle a_{lm'} a_{l-m'} \rangle = (2l+1)^2 C_l^2$$

$$\begin{aligned} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \langle a_{lm} a_{lm'} \rangle \langle a_{l-m} a_{l-m'} \rangle &= \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \langle a_{lm} a_{l-m'} \rangle \langle a_{l-m} a_{lm'} \rangle \\ &= (2l+1) C_l^2 \end{aligned}$$

Based on the above, we have

$$\langle (C_l^{\rm obs})^2 \rangle = \frac{C_l^2\left[(2l+1)^2 + 2(2l+1)\right]}{(2l+1)^2} = C_l^2\left[1 + \frac{2}{2l+1}\right],$$

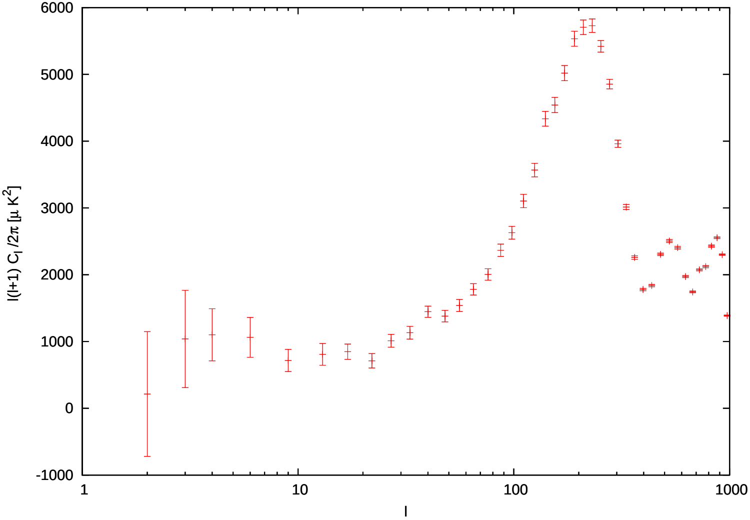
what gives

$$\left\langle \left(C_l - C_l^{\rm obs}\right)^2 \right\rangle = -C_l^2 + \langle (C_l^{\rm obs})^2 \rangle = C_l^2 \frac{2}{2l+1}.$$

Finally

Cosmic variance:

$$\frac{\Delta C_l}{C_l} := \frac{\sqrt{\left\langle \left(C_l - C_l^{\rm obs}\right)^2 \right\rangle}}{C_l} = \sqrt{\frac{2}{2l+1}}$$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